MATH 3403

TUTORIAL SHEET 9

1. Use the method of reflections to determine the Green's Function for the half-plane $y > 0$ in $\mathbb{R}^2$.

Hence show that the solution of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \; ; \; -\infty < x < \infty \; ; \; 0 < y < \infty$$
$$u(x, 0) = f(x)$$

is given by

$$u(\xi, \eta) = \frac{\eta}{\pi} \int_{-\infty}^{\infty} \frac{f(x)\, dx}{(x - \xi)^2 + \eta^2} \; .$$

2. Use the integral formula above to determine u when $f(x) = 1$, $x > 0$, $f(x) = 0$, $x < 0$.

3. In one dimension, Laplace's equation is

$$\frac{d^2 u}{dx^2} = 0 \; .$$

Show that
if $G(x, \xi)$ is a solution of this equation for $0 \le x < \xi \le 1$ such that $G(0, \xi) = 0$, then $G(x, \xi) = f(\xi)x$, and
if $G(x, \xi)$ is a solution of the equation for $0 \le \xi < x \le 1$ such that $G(1, \xi) = 0$, then $G(x, \xi) = g(\xi)(1 - x)$.
If, in addition, $G(x, \xi) = G(\xi, x)$ for $0 \le x \le 1$, $0 \le \xi \le 1$, show that

$$G(x, \xi) = \begin{cases} cx(1 - \xi) \; ; \; 0 \le x < \xi \le 1 \\ c\xi(1 - x) \; ; \; 0 \le \xi < x \le 1 \end{cases} \; .$$

Determine the constant c if the solution of the equation

$$\frac{d^2 u}{dx^2} = F(x) \; ; \; u(0) = u(1) = 0$$

is

$$u(x) = -\frac{1}{2} \int_0^1 G(x, \xi) F(\xi)\, d\xi \; .$$

(Hint: Choose $F(x) = 1$.)

4. Find the Fourier sine expansion

$$G = \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

for the function G in question 3.

Assignment Questions 1 and 2.