MATH3403

Tutorial Sheet 7

1. Show that

$$\frac{d}{dx}\left(x^{\nu}J_{\nu}(x)\right) = x^{\nu}J_{\nu-1}(x)\ ,$$

and that

$$\frac{d}{dx}\left(x^{-\nu}J_{\nu}(x)\right) = -x^{-\nu}J_{\nu+1}(x)\ .$$

Hence express $J_{\nu}'(x)$ in terms of $J_{\nu}(x)$ and $J_{\nu-1}(x)$, and also in terms of $J_{\nu}(x)$ and $J_{\nu+1}(x)$.

Combine these results to obtain a three term recurrence relation connecting $J_{\nu-1}(x)$, $J_{\nu}(x)$ and $J_{\nu+1}(x)$.

2. Show that $y = J_n(\lambda x)$ satisfies the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (\lambda^2x^2 - n^2)y = 0$$

Hence show that, for $\lambda \neq \mu$,

$$(\lambda^2 - \mu^2)\int_0^a xJ_n(\lambda x)J_n(\mu x)\,dx = a\left[\mu J_n(\lambda a)J_n'(\mu a) - \lambda J_n(\mu a)J_n'(\lambda a)\right]$$

Use L'Hôpital's rule to show that

$$\int_0^a x\left(J_n(\mu x)\right)^2\,dx = \frac{1}{2}a^2\left(J_n'(\mu a)\right)^2 + \frac{1}{2}\left(a^2 - \frac{n^2}{\mu^2}\right)\left(J_n(\mu a)\right)^2$$

3. Determine the solution of the cylindrical wave equation

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{\partial^2 u}{\partial t^2}$$
$$t \geq 0\ ;\ 0 \leq r < 1$$

with the initial conditions

$$u(r,0) = \epsilon(1 - r^2)\ ;\ u_t(r,0) = 0$$

and the boundary condition

$$u(1,t) = 0$$

Assignment Question 3