MATH 3403

TUTORIAL SHEET 10

1. Use Dirichlet's principle to obtain approximate solutions of the following equations, starting with the given functions.

(a)
$$u'' = xu, u(0) = 0, u(1) = 1;$$
$$w = x + ax(1 - x)$$

(b)
$$u'' = \sin x\, u, u(0) = u(\pi) = 1;$$
$$w = 1 + a \sin x$$

(c)
$$u_{xx} + u_{yy} = u, u(0, y) = u(x, 0) = 0, u(1, y) = y, u(x, 1) = x;$$
$$w = xy + a \sin \pi x \sin \pi y$$

(d)
$$u'' = u, u(0) = 0, u(1) = 1;$$
$$w = x + \sum_{n=1}^{\infty} a_n \sin n\pi x$$

2. By considering
$$f(x) = 1 - |x|^{\alpha}, \alpha > 1,$$
show that an approximate value for the first eigenvalue μ for
$$y'' + \mu^2 y = 0, y(-1) = y(1) = 0,$$
is $\mu_1 \simeq (\sqrt{2} + \sqrt{3})/2 \simeq 1.573$. (The exact value is $\pi/2 = 1.5708$.)

3. Let
$$\mathcal{D} = \{x, y; 0 < x < \pi, 0 < y < \pi\}$$
Find the solution in $\mathcal{D} \times \mathbb{R}^+$ of
$$u_{xx} + u_{yy} = u_{tt},$$
$$u(x, y, 0) = \sin^3 x \sin^3 y,$$
$$u_t(x, y, 0) = 0,$$
$$u(x, y, t) = 0 \text{ on } \partial\mathcal{D}.$$

Hint: $\sin^3 x = \frac{3}{4} \sin x - \frac{1}{4} \sin 3x$.

4. (a) If $P(\mathbf{x}) \geq 0$ on $\mathcal{D}$, show that
$$\nabla^2 \phi = \lambda P \phi \text{ in } \mathcal{D}, \phi = 0 \text{ on } \partial\mathcal{D}$$
has no non-trivial solution for $\lambda \geq 0$.

(b) If

$$\nabla^2\phi + \mu^2 P\phi = 0 \ ,$$

show that

$$\mu^2 \int_{\mathcal{D}} P\phi^2 = \int_{\mathcal{D}} (\nabla\phi)^2 \ .$$

(c) If

$$\nabla^2\phi_1 + \mu_1^2 P\phi_1 = 0 \ ,$$

and

$$\nabla^2\phi_2 + \mu_2^2 P\phi_2 = 0 \ ,$$
$$\mu_1^2 \ \neq \ \mu_2^2 \ ,$$

show that

(i) $$\int_{\mathcal{D}} P\phi_1\phi_2 = 0$$

(ii) $$\int_{\mathcal{D}} \nabla\phi_1\nabla\phi_2 = 0 \ .$$

(d) If $f = \sum_{n=1}^{\infty} a_n\phi_n$ show that

$$\frac{\int_{\mathcal{D}}(\nabla f)^2}{\int_{\mathcal{D}} Pf^2} \ \geq \ \mu_1^2,$$

where μ_1^2 is the smallest eigenvalue .

(e) Show that

$$\int_{\mathcal{D}} P(f - \sum_{i=1}^{n} a_i\phi_i)^2$$

is minimised by taking

$$a_i = \int_{\mathcal{D}} Pf\phi_i / \int_{\mathcal{D}} P\phi_i^2.$$

5. Let $\mathcal{D}$ be the unit circle.
By considering

$$f(r) = 1 - r^\alpha$$

show that an approximate value for the first eigenvalue for the equation

$$\nabla^2\phi + \mu^2 r\phi = 0 \text{ in } \mathcal{D},$$
$$\phi = 0 \text{ on } \partial\mathcal{D}$$

is $\mu_1 \ \simeq \ \frac{3}{2}(1+\sqrt{2}) \ \simeq \ 3.621$. (The exact value is 3.607 .)

Assignment. 1(b) and 5.