MA311

OPTIMAL ROUTE FOR A FERRY

Statement of the problem.

A ferry moves with constant speed from a terminal on one bank of a river to the other terminal directly opposite.

What course should it steer to make the journey in minimum time, taking account of the current?

Mathematical model.

Take the x-axis across the river and the y-axis down-stream.

Scale distance with respect to the width of the river, so that the ferry starts at $(0, 0)$ and finishes at $(1, 0)$.

Scale time so that the ferry can cross the river in unit time in the absence of any current. With this scaling the speed of the ferry is 1 unit.

As the control variable we take the angle θ between the course being steered and the x-axis.

If the current speed (in the y direction) is given by $f(x)$; $0 \leq x \leq 1$, then the velocity components of the ferry relative to fixed axes are

$$\dot{x} = \cos\theta$$
$$\dot{y} = \sin\theta + f(x)$$

Choosing x as the independent variable, the time taken to cross the river is

$$\int dt = \int_0^1 \frac{dx}{\dot{x}} = \int_0^1 \sec\theta \, dx$$

while the y displacement satisfies

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \tan\theta + f(x)\sec\theta \quad y(0) = y(1) = 0$$

The problem is therefore to minimise the first integral subject to the differential equation constraint.

Solution.

Taking $p(x)$ as the Lagrange multiplier, we obtain the Hamiltonian

$$\mathcal{H} = \sec\theta + p(x)(\tan\theta + f(x)\sec\theta)$$

Therefore

$$\frac{dp}{dx} = -\frac{\partial\mathcal{H}}{\partial y} = 0$$

so that p is a constant.[1]

$$\frac{\partial\mathcal{H}}{\partial\theta} = (1 + pf(x))\sec\theta\tan\theta + p\sec^2\theta = 0$$

$$\sin\theta = \frac{-p}{1 + pf(x)}$$

[1]Since the right hand side of the differential equation constraint is independent of y, we could replace it with the integral constraint $\int_0^1(\tan\theta + f(x)\sec\theta)\,dx = 0$

so that the value of p can be determined from the requirement that

$$\int_0^1 \frac{\sin\theta + f(x)}{\cos\theta}\, dx = 0$$

which gives

$$\int_0^1 \frac{(1+pf)f - p}{\sqrt{(1+pf)^2 - p^2}}\, dx = 0$$

In general this equation must be solved numerically.

Once the value of p is determined, the passage time and route can be found by numerical integration.