MA311

DISCRETE OPTIMAL CONTROL

Consider the discrete form of the optimal control problem.

We have a discrete cost function

$$J = F(x_N) + \sum_{k=0}^{N-1} f(x_k, u_k, k) \ ,$$

where $x_k \in \mathbb{R}^n$ is the state variable, and $u_k \in \mathbb{R}^m$ is the control variable.

The state and control variables are connected by the difference equation

$$x_{k+1} = g(x_k, u_k, k) \ ; \ x_0 = a \ .$$

We incorporate the difference equation into the cost function by introducing Lagrange multipliers (co-state variables) $p_k \in \mathbb{R}^n$ so that our modified cost function is

$$\begin{aligned}
\hat{J} &= F(x_N) + \sum_{k=0}^{N-1} \left[f(x_k, u_k, k) + p'_{k+1} \left(g(x_k, u_k, k) - x_{k+1} \right) \right] \\
&= F(x_N) - p'_N x_N + \sum_{k=1}^{N-1} \left[f(x_k, u_k, k) + p'_{k+1} g(x_k, u_k, k) - p'_k x_k \right] \\
&\quad + f(x_0, u_0, 0) + p'_1 g(x_0, u_0, 0)
\end{aligned}$$

If we introduce a small variation δu_k, $k = 0, \ldots, N-1$, in the controls, this will induce a small variation δx_k, $k = 1, \ldots, N$ in the state variables, and a first order variation in the modified cost function given by

$$\begin{aligned}
\delta \hat{J} = & \frac{dF}{dx}(x_N)\delta x_N - p'_N \delta x_N \\
& + \sum_{k=1}^{N-1} \left[\left(\frac{\partial f}{\partial x} + p'_{k+1} \frac{\partial g}{\partial x} - p'_k \right) \delta x_k \right. \\
& \left. \left(\frac{\partial f}{\partial u} + p'_{k+1} \frac{\partial g}{\partial u} \right) \delta u_k \right] \\
& \left(\frac{\partial f}{\partial u} + p'_1 \frac{\partial g}{\partial u} \right) \delta u_0
\end{aligned}$$

Provided u is unbounded, the optimum control occurs when the first variation $\delta \hat{J} = 0$. Hence the co-state variables satisfy the difference equation

$$p'_k = f_x(x_k, u_k, k) + p'_{k+1} g_x(x_k, u_k, k) \ ; \ p'_N = \nabla F(x_N)$$

and the optimum control satisfies

$$f_u(x_k, u_k, k) + p'_{k+1} g_u(x_k, u_k, k) = 0 \ ; \ k = 0, \ldots, N-1 \ .$$

The Linear - Quadratic Case

The equations take a simple form when the difference equation is linear in the state variables and controls

$$x_{k+1} = Ax_k + Bu_k$$

where A is a constant $n \times n$ matrix and B is a constant $n \times m$ matrix, and the cost function is quadratic in the state variables and controls

$$J = \frac{1}{2}x'_N S_N x_N + \sum_{k=0}^{N-1}\left[\frac{1}{2}x'_k Q x_k + \frac{1}{2}u'_k R u_k\right]$$

where S_N and Q are symmetric $n \times n$ matrices, and R is a positive definite symmetric $m \times m$ matrix.

Substituting these into the equations above gives the equations

$$\begin{aligned} p'_k &= x'_k Q + p'_{k+1}A \;; p'_N = x'_N S_N \\ p_k &= Qx_k + A'p_{k+1} \;; p_N = S_N x_N \end{aligned}$$

$$\begin{aligned} Ru_k + B'p_{k+1} &= 0 \\ u_k &= -R^{-1}B'p_{k+1} \end{aligned}$$

Substituting the last result into the original difference equation gives

$$x_{k+1} = Ax_k - BR^{-1}B'p_{k+1}$$

To solve the combined system we substitute $p_k = S_k x_k$, and solve for the unknown matrices S_k.

From the last equation we have

$$\begin{aligned} x_{k+1} &= Ax_k - BR^{-1}B'S_{k+1}x_{k+1} \\ \left(I + BR^{-1}B'S_{k+1}\right)x_{k+1} &= Ax_k \\ x_{k+1} &= \left(I + BR^{-1}B'S_{k+1}\right)^{-1}Ax_k \end{aligned}$$

so that the difference equation for the co-state variables becomes

$$\begin{aligned} S_k x_k &= Qx_k + A'S_{k+1}x_{k+1} \\ &= Qx_k + A'S_{k+1}\left(I + BR^{-1}B'S_{k+1}\right)^{-1}Ax_k \\ S_k &= Q + A'S^{k+1}\left(I + BR^{-1}B'S_{k+1}\right)^{-1}A \end{aligned}$$

This equation is known as a *discrete Ricatti equation*. It can be solved iteratively from $k = N$ to $k = 0$ using the known value S_N.

Once the matrices S_k are known the x_k can be determined from the equation

$$x_{k+1} = \left(I + BR^{-1}B'S_{k+1}\right)^{-1}Ax_k$$

and from these the controls follow.