MA311

DISCRETE VARIATION

DIDO'S PROBLEM

Discrete Euler Equations.

Consider the problem of optimising

$$J = \sum_{n=0}^{N-1} f(n, x_n, x_{n+1}) = \sum_{n=0}^{N-1} f(n, p, q) \ .$$

Suppose that the optimum solution is denoted by X_n, and let

$$x_n = X_n + \delta x_n \ .$$

Near the optimum we have the first variation

$$\begin{aligned} J &= \sum_{n=0}^{N-1} f(n, X_n, X_{n+1}) + f_p(n, X_n, X_{n+1})\delta x_n + f_q(n, X_n, X_{n+1})\delta x_{n+1} \\ &= \sum_{n=0}^{N-1} f(n, X_n, X_{n+1}) + f_p(0, X_0, X_1)\delta x_0 \\ &\quad + \sum_{n=1}^{N-1} \left(f_p(n, X_n, X_{n+1}) + f_q(n-1, X_{n-1}, X_n)\right)\delta x_n + f_q(N-1, X_{N-1}, X_N)\delta x_N \end{aligned}$$

The requirement that the first variation should vanish gives the recurrence relation

$$f_p(n, X_n, X_{n+1}) + f_q(n-1, X_{n-1}, X_n) = 0$$

for the optimum solution, together with the transversality conditions

$$\begin{aligned} f_p(0, X_0, X_1)\delta x_0 &= 0 \\ f_q(N-1, X_{N-1}, X_N)\delta x_N &= 0 \end{aligned}$$

Where x_0 is specified, the variation $\delta x_0 = 0$, and the transversality condition is satisfied. Similarly the condition at N is satisfied if x_N is specified.

If one or other of the end values is however not specified then the boundary condition for the difference equation is supplied by the equation

$$f_p(0, X_0, X_1) = 0$$

if x_0 is unknown, and by

$$f_q(N-1, X_{N-1}, X_N) = 0$$

when x_N is not given.

Example.

Minimise

$$J = \frac{1}{4} \sum_{n=0}^{9} 3x_n^2 - 2x_n x_{n+1} + x_{n+1}^2$$

(a) when $x_0 = x_{10} = 1$; (b) when $x_0 = 1$.

$$f(n,p,q) = \frac{1}{4}(3p^2 - 2pq + q^2) \ ; \ f_p = \frac{1}{2}(3p - q) \ ; \ f_q = \frac{1}{2}(-p + q)$$

The recurrence relation is

$$\frac{1}{2}(3x_n - x_{n+1}) + \frac{1}{2}(-x_{n-1} + x_n) = 0$$

$$x_{n+1} - 4x_n + x_{n-1} = 0$$

for which the solution is

$$x_n = A\alpha^n + B\alpha^{-n} \ ; \ \alpha = 2 + \sqrt{3}$$

In case (a), we have fixed endpoints, so that $\delta x_0 = \delta x_{10} = 0$. The constants A and B are determined from the given values.

$$\begin{aligned} A\alpha^0 + B\alpha^{-0} &= 1 \\ A\alpha^{10} + B\alpha^{-10} &= 1 \\ A &= 1/(1+\alpha^{10}) \\ B &= 1/(1+\alpha^{-10}) \\ x_n &= \frac{\alpha^n}{1+\alpha^{10}} + \frac{\alpha^{-n}}{1+\alpha^{-10}} \\ x_0 = x_{10} &= 1. \\ x_1 = x_9 &= 0.268 \\ x_2 = x_8 &= 0.072 \\ x_3 = x_7 &= 0.019 \\ x_4 = x_6 &= 0.0055 \\ x_5 &= 0.0028 \end{aligned}$$

In case (b), we have a 'free' choice of x_{10}, so that the transversality condition gives

$$f_q(9, x_9, x_{10}) \equiv \frac{1}{2}(-x_9 + x_{10}) = 0 \ ; \ x_9 = x_{10}$$

Since the recurrence relation is linear, we can use this condition to derive the solution backwards.

$$x_{10} = c, x_9 = c, x_8 = 4x_9 - x_{10} = 3c, x_7 = 11c, x_6 = 41c$$

$$x_5 = 153c, x_4 = 571c, x_3 = 2131c, x_2 = 7953c, x_1 = 29681c, x_0 = 110771c$$

$$c = 1/110771 \simeq .00001$$

Dido's Problem.

Determine the maximum area which can be enclosed by a curve of length 150, the x-axis and the line $x = 100$, if the curve begins at the origin.

Imagine that the curve is made up of 150 links of unit length; the nth link runs from (x_{n-1}, y_{n-1}) to (x_n, y_n).

The area enclosed between the nth link and the line $x = 100$, is

$$A_n = \frac{1}{2}(y_n - y_{n-1})(200 - (x_{n-1} + x_n))$$
$$= \frac{1}{2}\sqrt{1 - (x_n - x_{n-1})^2}(200 - (x_{n-1} + x_n))$$

The problem is therefore to maximise $J = \sum A_n$ subject to $x_0 = 0$ and $x_{150} = 100$.

The recurrence relation for the optimum solution is

$$\frac{1 - 2x_n^2 + 2x_n x_{n+1} + 200(x_n - x_{n+1})}{\sqrt{1 - (x_n - x_{n+1})^2}} = -\frac{1 - 2x_n^2 + 2x_n x_{n-1} + 200(x_n - x_{n-1})}{\sqrt{1 - (x_n - x_{n-1})^2}}$$

together with the **boundary** conditions on x_0 and x_{150}.

Note that since x_0 and x_{150} are fixed, the transversality conditions are satisfied with $\delta x_0 = 0$ and $\delta x_{150} = 0$.

Since the recurrence relation is non-linear, there is no simple general solution as in the previous example, and it is necessary to solve the problem numerically.

To this end we consider instead the following alternate problem:

Solve

$$\frac{1 - 2x_n^2 + 2x_n x_{n+1} + 200(x_n - x_{n+1})}{\sqrt{1 - (x_n - x_{n+1})^2}} = -\frac{1 - 2x_n^2 + 2x_n x_{n-1} + 200(x_n - x_{n-1})}{\sqrt{1 - (x_n - x_{n-1})^2}}$$

subject to the **initial** conditions $x_0 = 0$, $x_1 = \cos\theta$, where $0 < \theta < \pi/2$, and determine how many links are needed to reach the line $x = 100$.

By refining the choice of θ, we can determine the solution for which 150 links are needed.

Numerical considerations.

The recurrence relation has the terms

$$\sqrt{1 - (x_{n+1} - x_n)^2} \quad \text{and} \quad \sqrt{1 - (x_n - x_{n-1})^2}$$

in the denominators, so that the results may become 'inaccurate' as the required curve becomes horizontal towards the end.

To avoid this, set $a = 200 - 2x_n$ and $Dx_n = x_n - x_{n-1}$.

The recurrence relation becomes

$$\frac{1 - aDx_{n+1}}{\sqrt{1 - Dx_{n+1}^2}} = -\frac{1 + aDx_n}{\sqrt{1 - Dx_n^2}}$$

$$\frac{1 - 2aDx_{n+1} + a^2Dx_{n+1}^2}{1 - Dx_{n+1}^2} = \frac{1 + 2aDx_n + a^2Dx_n^2}{1 - Dx_n^2}$$

$$1 - Dx_n^2 - 2a(1 - Dx_n^2)Dx_{n+1} + a^2(1 - Dx_n^2)Dx_{n+1}^2 =$$
$$1 + 2aDx_n + a^2Dx_n^2 - (1 + 2aDx_n + a^2Dx_n^2)Dx_{n+1}^2$$

$$(a^2 + 2aDx_n + 1)Dx_{n+1}^2 - 2a(1 - Dx_n^2)Dx_{n+1} - (a^2Dx_n^2 + 2aDx_n + Dx_n^2) = 0 \;.$$

in which form the iterations are stable since the coefficient $a^2 + 2aDx_n + 1$ is ≥ 1 throughout the solution process.

Extensions.

Having found the solution for a curve of length 150, you can use the same program to find solutions for lengths of 110, 120, 130 and 140.

What limits are there on the length of the chain?