MA311

Tutorial Sheet 4

Solutions

1. Find the critical curves of the functional

$$J(y) = \int_0^1 ((y')^2 + x^2)\,dx\ ,\ y(0) = y(1) = 0$$

subject to the constraint

$$\int_0^1 y^2\,dx = 2\ .$$

Introducing the Lagrange multiplier p, we have the functional

$$\int_0^1 ((y')^2 + x^2 + py^2)\,dx\ .$$

The Euler-Lagrange equation for this functional is

$$\frac{d}{dx}(2y') = 2py$$
$$y'' = py$$

The boundary value problem

$$y'' = py\ ;\ y(0) = y(1) = 0\ ;$$

has non-trivial solutions of the form $y = a\sin(n\pi x)$ when $p = -n^2\pi^2$.
Introducing this form into the constraint, we obtain

$$a^2\int_0^1 \sin^2(n\pi x)\,dx = 2$$
$$\frac{a^2}{2}\int_0^1 (1-\cos(2n\pi x))\,dx = 2$$
$$a^2 = 4\ ;\ a = \pm 2$$

Therefore the critical curves are

$$y = \pm 2\sin(n\pi x)\ ;\ n = 1, 2, \dots\ .$$

2. Find the function which minimises the functional

$$J(y) = \int_0^1 (y')^2\,dx\ ,\ y(0) = 0\ ,\ y(1) = 1$$

with

$$\int_0^1 y\,dx = \frac{1}{3}\ .$$

Introducing the Lagrange multiplier p, we obtain

$$\int_0^1 ((y')^2 + py)\, dx \ .$$

The Euler-Lagrange equation is

$$\begin{aligned} \frac{d}{dx}(2y') &= p \\ 2y'' &= p \\ y &= a + bx + \frac{1}{4}px^2 \\ y(0) &= a = 0 \\ y(1) &= b + \frac{p}{4} = 1 \\ \int_0^1 y\, dx &= \frac{b}{2} + \frac{p}{12} = \frac{1}{3} \\ b = 0 \ ; p &= 4 \ ; y = x^2 \end{aligned}$$

3. Find the function y which maximises the integral

$$J(y) = -\int_{-\infty}^{\infty} y \log(y)\, dx \qquad (y \geq 0)$$

subject to the two constraints

$$\int_{-\infty}^{\infty} y\, dx = 1 \ , \ \int_{-\infty}^{\infty} x^2 y\, dx = \sigma^2 \ ,$$

where $y \to 0$ as $x \to \pm\infty$.

This problem arises in the theory of probability, where y is a probability density, and $J(y)$ is a quantity known as the entropy.

Introducing Lagrange multipliers p and q, we obtain

$$\int_{-\infty}^{\infty} \left(py + qx^2 y - y \log(y)\right)\, dx$$

Since the functional is independent of y', the Euler-Lagrange equation is simply

$$\begin{aligned} p + qx^2 - \log(y) - 1 &= 0 \\ y = e^{p-1} e^{qx^2} &= Ae^{-|q|x^2} \end{aligned}$$

since we require $q < 0$ in order to satisfy the boundary conditions on y.

Applying the constraints, we have

$$\begin{aligned}
A\int_{-\infty}^{\infty} e^{-|q|x^2}\,dx &= \frac{A}{\sqrt{|q|}}\int_{-\infty}^{\infty} e^{-t^2}\,dt = A\sqrt{\frac{\pi}{|q|}}\\
A\int_{-\infty}^{\infty} x^2 e^{-|q|x^2}\,dx &= \frac{A}{|q|\sqrt{|q|}}\int_{-\infty}^{\infty} t^2 e^{-t^2} = \frac{A}{2|q|}\sqrt{\frac{\pi}{|q|}}\\
A\sqrt{\frac{\pi}{|q|}} &= 1\\
\frac{A}{2|q|}\sqrt{\frac{\pi}{|q|}} &= \frac{1}{2|q|} = \sigma^2\\
A &= \frac{1}{\sigma\sqrt{2\pi}}\\
y &= \frac{1}{\sigma\sqrt{2\pi}}e^{-x^2/2\sigma^2}
\end{aligned}$$

which is the Gaussian distribution.

4. A particle of unit mass moves along the x-axis subject to a force $u(t)$. It is required to determine the control which transfers the particle from rest at the origin to rest at $x = 1$ in unit time, so as to minimise the effort involved, measured by

$$\int_0^1 u^2\,dt\ .$$

Newton's Second Law gives the equation

$$\ddot{x} = u$$

which can be written as a pair of first order equations

$$\dot{x} = v\ ;\ \dot{v} = u$$

subject to the boundary conditions

$$u(0) = v(0) = v(1) = 0\ ;\ u(1) = 1\ .$$

Introducing Lagrange multipliers p and q, we obtain the Hamiltonian

$$\mathcal{H} = u^2 + pv + qu$$

from which we derive the equations

$$\begin{aligned}
\dot{p} &= -\frac{\partial\mathcal{H}}{\partial x} = 0\\
\dot{q} &= -\frac{\partial\mathcal{H}}{\partial v} = -p\\
0 &= \frac{\partial\mathcal{H}}{\partial u} = 2u + q
\end{aligned}$$

Solving the systems we obtain

$$\begin{aligned} p &= p_0 \\ q &= q_0 - p_0 t \\ u &= \frac{p_0}{2}t - \frac{q_0}{2} \\ v &= v_0 - \frac{q_0}{2}t + \frac{p_0}{4}t^2 \\ x &= x_0 + v_0 t - \frac{q_0}{4}t^2 + \frac{p_0}{12}t^3 \end{aligned}$$

and using the boundary conditions we get

$$\begin{aligned} x_0 = v_0 &= 0 \\ \frac{p_0}{4} - \frac{q_0}{2} &= 0 \\ \frac{p_0}{12} - \frac{q_0}{4} &= 1 \\ q_0 = -12 \ &; \ p_0 = -24 \\ u &= 6 - 12t \end{aligned}$$

5. Consider the system

$$\dot{x}_1 = x_2 \quad , \quad \dot{x}_2 = -x_1 + u \quad .$$

Determine the control $u(t)$ which transfers it from $x(0) = 0$ to the line $x_1 + 5x_2 = 15$ when $t = 2$, and which minimises

$$\frac{1}{2}\left[x_1(2) - 5\right]^2 + \frac{1}{2}\left[x_2(2) - 2\right]^2 + \frac{1}{2}\int_0^2 u^2 \, dt \ .$$

Let p_1 and p_2 be the Lagrange multipliers.

While we have two initial conditions specified; $x_1(0) = x_2(0) = 0$; we have only one terminal condition. The second terminal condition must therefore be supplied by the transversality conditions.

If the terminal point lies on the line $x_1 + 5x_2 = 15$, then

$$\delta x_1(2) + 5\delta x_2(2) = 0 \ ,$$

while the transversality requires

$$\begin{aligned} (x_1(2) - 5)\delta x_1(2) + (x_2(2) - 2)\delta x_2 - p_1(2)\delta x_1(2) - p_2(2)\delta x_2(2) &= 0 \\ (-5x_1(2) + 25 + x_2 - 2 + 5p_1(2) - p_2(2))\delta x_2(2) &= 0 \\ 5x_1(2) - x_2(2) - 5p_1(2) + p_2(2) &= 23 \end{aligned}$$

The Hamiltonian for the problem is

$$\mathcal{H} = \frac{1}{2}u^2 + p_1 x_2 + p_2(u - x_1) \ .$$

The equations are

$$\dot{p_1} = -\frac{\partial \mathcal{H}}{\partial x_1} = p_2$$
$$\dot{p_2} = -\frac{\partial \mathcal{H}}{\partial x_2} = -p_1$$
$$0 = \frac{\partial \mathcal{H}}{\partial u} = u + p_2$$
$$p_1 = A\cos t + B\sin t$$
$$p_2 = -A\sin t + B\cos t$$
$$u = A\sin t - B\cos t$$
$$x_1 = C\sin t + D\cos t - \frac{1}{2}At\cos t - \frac{1}{2}Bt\sin t$$
$$x_2 = C\cos t - D\sin t - \frac{A}{2}(\cos t - t\sin t) - \frac{B}{2}(\sin t + t\cos t)$$

Applying the boundary conditions gives

$$D = 0$$
$$C - \frac{1}{2}A = 0 \ ; \ A = 2C$$
$$(C\sin 2 - 2C\cos 2 + B\sin 2) + 5(2C\sin 2 + \frac{B}{2}\sin 2 + B\cos 2) = 15$$
$$(11\sin 2 - 2\cos 2)C + (3.5\sin 2 + 5\cos 2)B = 15$$
$$5(C\sin 2 - 2C\cos 2 + B\sin 2) - (2C\sin 2 + \frac{B}{2}\sin 2 + B\cos 2)$$
$$-5(2C\cos 2 + B\sin 2) + (-2C\sin 2 + B\cos 2) = 23$$
$$(\sin 2 - 20\cos 2)C - 0.5\sin 2B = 23$$

The solution of these two simultaneous equations gives the required result.