

**MA311**  
TUTORIAL SHEET 3

1. Minimise

$$J = \sum_{n=0}^2 (x_{n+1}^2 - x_{n+1}x_n + x_n^2)$$

The Euler equations for this system are

$$\begin{aligned} -x_{n+1} + 2x_n + 2x_n - x_{n-1} &= 0 \\ x_{n+1} - 4x_n + x_{n-1} &= 0 \end{aligned}$$

(a) when  $x_0 = x_3 = 1$  ;

Since we have fixed endpoint values, the transversality conditions are satisfied.  
The general solution of the Euler equations is

$$x_n = A\alpha^n + B\alpha^{-n}$$

where

$$\alpha = 2 + \sqrt{3} .$$

Using the boundary conditions,

$$\begin{aligned} A\alpha^0 + B\alpha^{-0} &= 1 \\ A\alpha^3 + B\alpha^{-3} &= 1 \\ A &= \frac{1}{1 + \alpha^3} \\ B &= \frac{\alpha^3}{1 + \alpha^3} \\ x_n &= \frac{\alpha^n + \alpha^{3-n}}{1 + \alpha^3} \\ x_0 = x_3 &= 1 \\ x_1 = x_2 &= \frac{\alpha}{1 - \alpha + \alpha^2} = \frac{1}{3} \end{aligned}$$

(b) when  $x_0 = 1$ .

Since  $x_3$  is free, the terminal condition is

$$2x_3 - x_2 = 0 .$$

Setting  $x_3 = c$ , we have

$$\begin{aligned} x_2 &= 2c \\ x_1 &= 4x_2 - x_3 = 7c \\ x_0 &= 4x_1 - x_2 = 26c (= 1) \\ c &= \frac{1}{26} \end{aligned}$$

2. Derive the Euler equations associated with the problem of minimising

$$J = \sum_{n=0}^{N-1} \frac{x_n + x_{n+1}}{1 + x_n^2 + x_{n+1}^2}$$

subject to  $x_0 = 0$ ,  $x_N = 1$ .

The Euler equations are

$$\frac{(1 + x_n^2 + x_{n+1}^2) - 2x_n(x_n + x_{n+1})}{(1 + x_n^2 + x_{n+1}^2)^2} + \frac{(1 + x_{n-1}^2 + x_n^2) - 2x_n(x_n + x_{n-1})}{(1 + x_{n-1}^2 + x_n^2)^2} = 0.$$

What is the solution when  $N = 2$  ?

Substituting  $x_0 = 0$  and  $x_2 = 1$  into the Euler equation with  $n = 1$ , gives

$$\begin{aligned} \frac{2 - 2x_1 - x_1^2}{(2 + x_1^2)^2} + \frac{1 - x_1^2}{(1 + x_1^2)^2} &= 0 \\ (2 - 2x_1 - x_1^2)(1 + 2x_1^2 + x_1^4) + (1 - x_1^2)(4 + 4x_1^2 + x_1^4) &= 0 \\ 6 - 2x_1 + 3x_1^2 - 4x_1^3 - 3x_1^4 - 2x_1^5 - 2x_1^6 &= 0 \\ x_1 = 0.8820262046 \text{ or } x_1 = -1.614744838 \end{aligned}$$

Of these two values, the negative one obviously gives the minimum value of  $J$ ,

$$J = -0.5810455$$

while the positive value maximises  $J$ ;

$$J = 1.1735685$$