MA311

TUTORIAL SHEET 2

SOLUTIONS

1. Find the critical curves of the functionals

(i) $$\int_a^b (y^2 + (y')^2 - 2y \sin x)\, dx$$

$$\begin{aligned} f(x, y, y') &= y^2 + (y')^2 - 2y \sin x \\ \frac{\partial f}{\partial y} &= 2y - 2 \sin x \\ \frac{\partial f}{\partial y'} &= 2y' \\ \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) &= 2y'' \end{aligned}$$

The Euler equations are

$$\begin{aligned} 2y'' &= 2y - 2 \sin x \\ y'' - y &= -\sin x \end{aligned}$$

This is a linear equation, with constant coefficients. The general solution is the sum of a particular solution of the equation and the complementary function.

The complementary function is the general solution of the equation

$$y'' - y = 0$$

The characteritic equation associated with this equation is

$$m^2 - 1 = 0$$

whose roots are $m = 1$ *and* $m = -1$*. Therefore the complementary function is*

$$y_c = Ae^x + Be^{-x} \; .$$

To determine a particular solution, we use the method of undetermined coefficients.

Try

$$\begin{aligned} y_p &= a \sin + b \cos x \\ y_p'' &= -a \sin x - b \cos x \\ y_p'' - y_p &= -2a \sin x - 2b \cos x \\ \textit{which equals} \quad -\sin x \quad & \textit{when} \quad a = \frac{1}{2}\ ,\ b = 0 \end{aligned}$$

Therefore the general solution of the Euler equation is

$$y = Ae^x + Be^{-x} + \frac{1}{2}\sin x \ .$$

(ii)
$$\int_a^b (y^2 + (y')^2 + 2ye^x)\, dx$$

$$\begin{aligned} f(x,y,y') &= y^2 + (y')^2 + 2ye^x \\ \frac{\partial f}{\partial y} &= 2y + 2e^x \\ \frac{\partial f}{\partial y'} &= 2y' \\ \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) &= 2y'' \end{aligned}$$

The Euler equation is

$$y'' - y = e^x$$

The complementary function is the same as in the first part.

Since the right hand side, e^x, occurs as part of the complementary function, we look for a particular solution in the form

$$\begin{aligned} y_p &= axe^x \\ y_p' &= ae^x + axe^x \\ y_p'' &= 2ae^x + axe^x \\ y_p'' - y_p &= 2ae^x \\ \textit{which equals} \quad e^x \quad \textit{when} \quad a &= \frac{1}{2} \end{aligned}$$

Therefore the general solution of the Euler equation is

$$y = Ae^x + Be^{-x} + \frac{1}{2}xe^x \ .$$

2. Determine the smooth function $x(t)$ satisfying $x(0) = 0$, $x(1) = 1$ which minimises

$$J(x) = \int_0^1 \left(x^2 + (\dot{x} + x)^2\right)\, dt \quad .$$

$$\begin{aligned} f(t,x,\dot{x}) &= 2x^2 + 2x\dot{x} + \dot{x}^2 \\ \frac{\partial f}{\partial x} &= 4x + 2\dot{x} \\ \frac{\partial f}{\partial \dot{x}} &= 2x + 2\dot{x} \\ \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{x}}\right) &= 2\dot{x} + 2\ddot{x} \end{aligned}$$

The Euler equation is

$$\ddot{x} = 2x$$

for which the general solution is

$$x = Ae^{\sqrt{2}t} + Be^{-\sqrt{2}t} \ .$$

Applying the boundary conditions, we obtain

$$\begin{aligned} A + B &= 0 \\ Ae^{\sqrt{2}} + Be^{-\sqrt{2}} &= 1 \\ A\left(e^{\sqrt{2}} - e^{-\sqrt{2}}\right) &= 1 \\ A = \frac{1}{2\sinh(\sqrt{2})} \ ; \ B &= -\frac{1}{2\sinh(\sqrt{2})} \\ x &= \frac{\sinh(\sqrt{2}t)}{\sinh(\sqrt{2})} \end{aligned}$$

3. Find the critical curves of the functional

$$J(y,z) = \int_0^{\pi/2} \left((y')^2 + (z')^2 + 2yz\right)\, dx$$

subject to the boundary conditions

$$y(0) = 0 \ , \ y(\pi/2) = 1 \ , \ z(0) = 0 \ , \ z(\pi/2) = 1 \ .$$

$$\begin{aligned} \frac{\partial f}{\partial y} &= 2z \\ \frac{\partial f}{\partial y'} &= 2y' \\ \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) &= 2y'' \\ \frac{\partial f}{\partial z} &= 2y \\ \frac{\partial f}{\partial z'} &= 2z' \\ \frac{d}{dx}\left(\frac{\partial f}{\partial z'}\right) &= 2z'' \end{aligned}$$

The Euler equations are

$$\begin{aligned} y'' &= z \\ z'' &= y \end{aligned}$$

Eliminating z gives

$$y'''' = z'' = y$$

for which the general solution is

$$y = A\cosh x + B\sinh x + C\cos x + D\sin x$$

Since

$$\begin{aligned} z &= y'' \\ z &= A\cosh x + B\sinh x - C\cos x - D\sin x \end{aligned}$$

From the initial conditions

$$\begin{aligned} A + C &= 0 \\ A - C &= 0 \\ A = C &= 0 \end{aligned}$$

and from the terminal conditions

$$\begin{aligned} B\sinh\left(\frac{\pi}{2}\right) + D &= 1 \\ B\sinh\left(\frac{\pi}{2}\right) - D &= 1 \\ B = \frac{1}{\sinh(\pi/2)}\ &;\ D = 0 \end{aligned}$$

Therefore

$$y = z = \frac{\sinh(x)}{\sinh(\pi/2)}\ .$$

What happens if you replace $\pi/2$ by π throughout?

The general solution is unaffected, and the initial conditions are also unchanged. However, the terminal conditions now become

$$\begin{aligned} B\sinh(\pi) &= 1 \\ B\sinh(\pi) &= 1 \\ B = \frac{1}{\sinh(\pi)}\ &;\ D \textit{ undetermined} \end{aligned}$$

so that the solution is no longer unique.

$$\begin{aligned} y &= \frac{\sinh(x)}{\sinh(\pi)} + D\sin x \\ z &= \frac{\sinh(x)}{\sinh(\pi)} - D\sin x \end{aligned}$$

where D is arbitrary.

4. Find the general solution of the Euler equation for the functional

$$\int_a^b f(x)\sqrt{1+(y')^2}\,dx$$

$$\frac{\partial}{\partial y}\left(f(x)\sqrt{1+(y')^2}\right) = 0$$

$$\frac{\partial}{\partial y'}\left(f(x)\sqrt{1+(y')^2}\right) = f(x)\frac{y'}{\sqrt{1+(y')^2}}$$

The Euler equation has the form

$$\frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

for which a first integral is

$$\frac{\partial F}{\partial y'} = c$$

Therefore, in this case we have

$$\begin{aligned} f(x)\frac{y'}{\sqrt{1+(y')^2}} &= c \\ \frac{(y')^2}{1+(y')^2} &= \frac{c^2}{f^2} \\ (y')^2\left(1-\frac{c^2}{f^2}\right) &= \frac{c^2}{f^2} \\ y' &= \frac{c}{\sqrt{f^2-c^2}} \end{aligned}$$

provided $f(x) > c \geq 0$.

$$y = a + \int^x \frac{c}{\sqrt{f^2(s)-c^2}}\,ds$$

and investigate the special cases

(i) $$f(x) = \sqrt{x}$$

$$\begin{aligned} y &= a + \int_{c^2}^x \frac{c}{\sqrt{s-c^2}}\,ds \\ &= a + 2c\sqrt{x-c^2}\ ;\ x > c^2 \end{aligned}$$

(ii) $$f(x) = x$$

$$\begin{aligned} y &= a + \int_c^x \frac{c}{\sqrt{s^2-c^2}}\,ds \\ &= a + c\,\mathrm{arcosh}\left(\frac{x}{c}\right)\ ;\ x > c > 0 \end{aligned}$$