MA311

TUTORIAL SHEET 1

SOLUTIONS

1. Determine the point $\boldsymbol{v}$ which minimizes

$$\sum_{i=1}^{4} |\boldsymbol{v} - \boldsymbol{v}_i|$$

The Euler equations for this function are

$$\sum_{i=1}^{4} \frac{(\boldsymbol{v} - \boldsymbol{v}_i)}{|\boldsymbol{v} - \boldsymbol{v}_i|} = \mathbf{0} \ .$$

These are unit vectors. If four unit vectors in a plane sum to $\mathbf{0}$*, then they jointly form a rhombus, and therefore occur in equal and opposite pairs.*

(a) when the quadrilateral $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \boldsymbol{v}_4\}$ is convex;

In this case the previous condition is satisfied by the point of intersection of the diagonals of the quadrilateral.

(b) when the quadrilateral has a re-entrant angle.

In this case, there is no point which satisfies the Euler equations. However, the vertices are critical points, (at which the derivative is not defined), and the minimum occurs when $\boldsymbol{v}$ *is the re-entrant vertex.*

2. For the triangle with vertices $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ and $\boldsymbol{v}_3$, and side vectors

$$\boldsymbol{s}_i = \boldsymbol{v}_{i+1(\mod 3)} - \boldsymbol{v}_i$$

show that the square of the distance of a point $\boldsymbol{v}$ from the sides is

$$[(\boldsymbol{v} - \boldsymbol{v}_i) \times \hat{\boldsymbol{s}}_i]^2 = (\boldsymbol{v} - \boldsymbol{v}_i)^2 - [(\boldsymbol{v} - \boldsymbol{v}_i).\hat{\boldsymbol{s}}_i]^2 \ .$$

The projection of the vector $\boldsymbol{v} - \boldsymbol{v}_i$ *onto* $\boldsymbol{s}_i$ *is*

$$((\boldsymbol{v} - \boldsymbol{v}_i).\hat{\boldsymbol{s}}_i)\, \hat{\boldsymbol{s}}_i$$

the square of whose length is

$$((\boldsymbol{v} - \boldsymbol{v}_i).\hat{\boldsymbol{s}_i})^2 = (\boldsymbol{v} - \boldsymbol{v}_i)^2 \cos^2\theta$$

By Pythagoras' theorem, the square of the perpendicular distance from $\boldsymbol{v}$ *to the side is*

$$(\boldsymbol{v} - \boldsymbol{v}_i)^2 \sin^2\theta$$
$$= (\boldsymbol{v} - \boldsymbol{v}_i)^2 - ((\boldsymbol{v} - \boldsymbol{v}_i).\hat{\boldsymbol{s}_i})^2$$

Alternatively, since

$$|(\boldsymbol{v} - \boldsymbol{v}_i) \times \hat{\boldsymbol{s}_i}| = |\boldsymbol{v} - \boldsymbol{v}_i| \sin\theta$$

the square of this vector also provides an answer.

Show that if the point $\boldsymbol{v}$ is to minimise the sum of these squares, then the normal vectors from the point to the sides of the triangle sum to zero.

Suppose that $\boldsymbol{v}$ minimises

$$\sum_{i=1}^{3}(\boldsymbol{v}-\boldsymbol{v}_i)^2-((\boldsymbol{v}-\boldsymbol{v}_i).\hat{\boldsymbol{s}}_i)^2$$

Then for any $\boldsymbol{\eta}$,

$$f(\epsilon)=\sum_{i=1}^{3}(\boldsymbol{v}+\epsilon\boldsymbol{\eta}-\boldsymbol{v}_i)^2-((\boldsymbol{v}+\epsilon\boldsymbol{\eta}-\boldsymbol{v}_i).\hat{\boldsymbol{s}}_i)^2$$

has a minimum when $\epsilon=0$.

Differentiating with respect to ϵ, and setting $f'(\epsilon)=0$ when $\epsilon=0$, gives

$$\sum_{i=1}^{3}2(\boldsymbol{v}-\boldsymbol{v}_i).\boldsymbol{\eta}-2\left((\boldsymbol{v}-\boldsymbol{v}_i).\hat{\boldsymbol{s}}_i\right)(\hat{\boldsymbol{s}}_i.\boldsymbol{\eta})=0$$

Since $\boldsymbol{\eta}$ is arbitrary, this gives

$$\sum_{i=1}^{3}\left((\boldsymbol{v}-\boldsymbol{v}_i)-((\boldsymbol{v}-\boldsymbol{v}_i).\hat{\boldsymbol{s}}_i)\,\hat{\boldsymbol{s}}_i\right)=\mathbf{0}$$

as required.

What condition must be satisfied if the sum of the distances is to be minimized?

We wish to minimise

$$\sum_{i=1}^{3}|(\boldsymbol{v}-\boldsymbol{v}_i)\times\hat{\boldsymbol{s}}_i|$$

Once again, assume that $\boldsymbol{v}$ is the required vector, and consider

$$f(\epsilon)=\sum_{i=1}^{3}|(\boldsymbol{v}+\epsilon\boldsymbol{\eta}-\boldsymbol{v}_i)\times\hat{\boldsymbol{s}}_i|$$

Differentiating and setting $f'(0)=0$ gives

$$\sum i=1^3\frac{(\boldsymbol{v}-\boldsymbol{v}_i)\times\hat{\boldsymbol{s}}_i).(\boldsymbol{\eta}\times\hat{\boldsymbol{s}}_i)}{|(\boldsymbol{v}-\boldsymbol{v}_i)\times\hat{\boldsymbol{s}}_i|}=0$$

$$\begin{aligned}(\boldsymbol{v}-\boldsymbol{v}_i)\times\hat{\boldsymbol{s}}_i).(\boldsymbol{\eta}\times\hat{\boldsymbol{s}}_i)&=\boldsymbol{\eta}.\left(\hat{\boldsymbol{s}}_i\times((\boldsymbol{v}-\boldsymbol{v}_i)\times\hat{\boldsymbol{s}}_i)\right)\\&=\boldsymbol{\eta}.\left((\boldsymbol{v}-\boldsymbol{v}_i)-((\boldsymbol{v}-\boldsymbol{v}_i).\hat{\boldsymbol{s}}_i)\hat{\boldsymbol{s}}_i\right)\end{aligned}$$

so that the Euler condition is

$$\sum_{i=1}^{3}\frac{(\boldsymbol{v}-\boldsymbol{v}_i)-((\boldsymbol{v}-\boldsymbol{v}_i).\hat{\boldsymbol{s}}_i)\hat{\boldsymbol{s}}_i)}{|(\boldsymbol{v}-\boldsymbol{v}_i)\times\hat{\boldsymbol{s}}_i|}=\mathbf{0}$$

i.e. The three unit normals at $\boldsymbol{v}$ sum to $\mathbf{0}$.

This condition is only ever satisfied if the triangle is equilateral. Then it is satisfied by every point $\boldsymbol{v}$ inside the triangle.

3. Determine the solution of the problem of minimising

$$|\boldsymbol{v} - \boldsymbol{v}_1|^k + |\boldsymbol{v} - \boldsymbol{v}_2|^k \ ; \ k > 0$$

The Euler equations for this problem are

$$|\boldsymbol{v} - \boldsymbol{v}_1|^{k-2}(\boldsymbol{v} - \boldsymbol{v}_1) + |\boldsymbol{v} - \boldsymbol{v}_2|^{k-2}(\boldsymbol{v} - \boldsymbol{v}_2) = \mathbf{0}$$

For $k > 1$, the solution is the midpoint of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$.

When $k = 1$, any point on the line segment between $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ will do.

For $0 < k < 1$, the midpoint is again the solution of the Euler equations, but in this case it gives a maximum! The minimum occurs at either of the other critical points; i.e. at either $\boldsymbol{v}_1$ or $\boldsymbol{v}_2$.