MP213
SOLVED EXAMPLE 3
1995

ANATOMY OF A DIFFERENTIAL EQUATION

These notes represent a detailed discussion of the differential equation

$$(1-x^2)y'' - xy' + \alpha^2 y = 0$$

and its solutions.

This equation is a second order homogeneous linear equation. For such an equation written in the normal form

$$y'' + p(x)y' + q(x)y = 0$$

the existence and uniqueness of solutions is guaranteed provided the coefficient functions p and q are continuous.

In this case we have

$$p(x) = \frac{x}{x^2-1} \quad \text{and} \quad q(x) = \frac{\alpha^2}{1-x^2} ,$$

which are continuous for all x except $x = \pm 1$. These two points are the (finite) singular points of the equation.

In order to solve the equation fully, we need to consider each of the intervals $(-\infty, -1)$, $(-1, 1)$ and $(1, \infty)$.

For each of these intervals we need to find two independent solutions, generically referred to as y_1 and y_2 (though these names may be applied to different solutions on the different intervals). Because the equation is linear, these solution functions will exist throughout their appropriate interval, and the general solution of the differential equation on that interval has the form

$$y = ay_1 + by_2 .$$

This means that if we should find a third solution y_3 on the same interval, there must be constants A and B such that

$$y_3 = Ay_1 + By_2 .$$

Solutions on $(-1, 1)$

Since the origin is an ordinary point of the equation, we can determine two independent solutions of the equation in the form of power series:

$$y = \sum_{n=0}^{\infty} a_n x^n .$$

Because of the singular points at ± 1, these series will converge for $|x| < 1$.

Assuming the power series form for y, we have

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad ; \quad xy' = \sum_{n=0}^{\infty} n a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} \ = \ \sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n$$

$$x^2 y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^n$$

Notice that only expressions which appear in the differential equation need to be considered. For each of these, the terms of the sum have been expressed in the form $c_n x^n$, where c_n does not involve x. This required a rescaling only in the case of y''.

Substituting these expansions into the differential equation gives

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n - \sum_{n=0}^{\infty} (n-1) n a_n x^n - \sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} \alpha^2 a_n x^n = 0$$

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n = \sum_{n=0}^{\infty} ((n-1)n + n - \alpha^2) a_n x^n$$

and equating the coefficients of x^n we obtain the recurrence relation

$$(n+1)(n+2) a_{n+2} = (n^2 - \alpha^2) a_n \ .$$

Since this recurrence relation expresses a_{n+2} in terms of a_n without reference to the value of a_{n+1}, it is possible to find two solutions, one of which is an even function of x;

$$y_1 = \sum_{m=0}^{\infty} a_{2m} x^{2m}$$

and the other of which is an odd function of x;

$$y_2 = \sum_{m=0}^{\infty} a_{2m+1} x^{2m+1} \ .$$

The factor $(n^2 - \alpha^2)$, which vanishes when $\alpha = n$, shows that for integer values of the parameter α there is a polynomial solution of the differential equation.

Notice that $y_1(0) = a_0$, $y_1'(0) = 0$, and $y_2(0) = 0$, $y_2'(0) = a_1$, giving the Wronskian of these two solutions at $x = 0$ as

$$\begin{vmatrix} a_0 & 0 \\ 0 & a_1 \end{vmatrix} = a_0 a_1 \ ,$$

so that (provided neither a_0 nor a_1 is zero, which would correspond to an identically zero solution) these two solutions are independent solutions of the equation on

$(-1, 1)$ as required. In particular, if we take $a_0 = 1$ and $a_1 = 1$, the solution $Y(x)$ for which $Y(0) = A$, $Y'(0) = B$ is

$$Y(x) = Ay_1(x) + By_2(x) \ .$$

We can find the Wronskian of y_1 and y_2 on the interval by using Abel's Identity

$$W[y_1, y_2](x) = W[y_1, y_2](0) \exp\left(-\int_0^x p(s)\, ds \right)$$
$$\int_0^x p(s)\, ds = \int_0^x \frac{s}{s^2 - 1}\, ds = \frac{1}{2} \log(1 - x^2)$$
$$W[y_1, y_2] = \frac{a_1 a_2}{\sqrt{1 - x^2}}$$

(Remember that $\int dt/t = \log |t|$, which gives $\log(1-x^2)$ in this case since $|x| < 1$.)

Notice that the Wronskian becomes infinite at the endpoints of the interval. This signals the breakdown of the existence and uniqueness properties at these points.

Returning to the solution y_1, we make the substitution $n = 2m$ in the recurrence relation for the coefficients derived above. We obtain

$$(2m + 1)(2m + 2)a_{2(m+1)} = (4m^2 - \alpha^2)a_{2m} \ .$$

The ratio test tells us that the series

$$\sum_{m=0}^{\infty} a_{2m} x^{2m}$$

converges provided

$$\lim_{m \to \infty} \left| \frac{a_{2(m+1)} x^{2(m+1)}}{a_{2m} x^{2m}} \right| < 1 \ .$$

From the recurrence relation we have

$$\left| \frac{a_{2(m+1)}}{a_{2m}} \right| = \left| \frac{4m^2 - \alpha^2}{4m^2 + 6m + 2} \right| \ \to \ 1 \text{ as } m \to \infty$$

so that the series converges for $|x| < 1$, confirming our earlier remarks.

This test could be applied to the series for y_2 with similar results. Note that we can apply this test without knowing the exact forms for the coefficients a_n.

Solving the recurrence relation successively we obtain

$$\begin{aligned}
a_2 &= \frac{-\alpha^2}{1.2} a_0 \\
a_4 &= \frac{(4 - \alpha^2)}{3.4} a_2 \ = \ \frac{(-\alpha^2)(4 - \alpha^2)}{1.2.3.4} a_0 \\
a_{2m} &= \frac{(-\alpha^2)(4 - \alpha^2) \ldots (4(m-1)^2 - \alpha^2)}{1.2.3.4. \ \ldots \ (2m-1).(2m)} a_0 \\
&= \frac{1}{(2m)!} \prod_{r=0}^{m-1} (4r^2 - \alpha^2)\, a_0 \ ,
\end{aligned}$$

so that

$$y_1 = a_0 \left(1 + \sum_{m=1}^{\infty} \left(\prod_{r=0}^{m-1} (4r^2 - \alpha^2) \right) \frac{x^{2m}}{(2m)!} \right) . \tag{1}$$

If $\alpha = 2N$, where N is an integer, the factor $(4r^2 - \alpha^2)$ becomes 0 when $r = N$, so that the product vanishes when $m - 1 \geq N$; that is, from $m = N + 1$ onwards. The last non-zero term in the expansion is

$$\begin{aligned} a_{2N} &= \frac{a_0}{(2N)!}(-4)^N \prod_{r=0}^{N-1} (N^2 - r^2) \\ &= (-4)^N a_0 \frac{1.2.\ldots.(N-1).N.N.(N+1).\ldots.(2N-1)}{(2N)!} \\ &= (-1)^N 2^{2N-1} a_0 \text{ for } N > 0 \ . \end{aligned}$$

and the corresponding solution is a polynomial of degree $2N$ in x.

When suitably scaled, this polynomial is called a Chebychev polynomial, and is usually designated by $T_{2N}(x)$. The scaling is chosen so that $T_n(1) = 1$. For the even polynomials which we have just derived, this scaling amounts to choosing $a_0 = (-1)^N$. This in turn gives $a_{2N} = 2^{2N-1}$ for $N > 0$.

The first few polynomials of this type are

$$\begin{aligned} T_0(x) &= 1 \\ T_2(x) &= 2x^2 - 1 \\ T_4(x) &= 8x^4 - 8x^2 + 1 \end{aligned}$$

In order to determine the second solution in odd powers of x, we set $n = 2m+1$ in the recurrence relation, obtaining

$$(2m+2)(2m+3)a_{2(m+1)+1} = ((2m+1)^2 - \alpha^2)a_{2m+1}$$

and proceding as before we derive

$$\begin{aligned} a_{2m+1} &= \frac{(1-\alpha^2)(9-\alpha^2)\ldots((2m-1)^2-\alpha^2)}{2.3.4.\ldots(2m)(2m+1)} a_1 \\ &= \frac{1}{(2m+1)!} \prod_{r=0}^{m-1} ((2r+1)^2 - \alpha^2) a_1 \end{aligned}$$

so that

$$y_2 = a_1 \left(x + \sum_{m=1}^{\infty} \left(\prod_{r=0}^{m-1} ((2r+1)^2 - \alpha^2) \right) \frac{x^{2m+1}}{(2m+1)!} \right) . \tag{2}$$

(When $\alpha = 0$, this solution has the form

$$y_2 = a_1 \left(x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \ldots \right)$$

which is the Taylor Series expansion of a function which you have met in first year. You may care to look up some reference to find out what this function is. Alternatively, you could derive the closed form by considering the Wronskian of y_1 and y_2 when $\alpha = 0$.

Similarly, when $\alpha = 1$, the solution y_1 can be written as

$$y_1 = a_0 \left(1 + \frac{1}{2} \frac{(-x^2)}{1!} + \frac{1}{2} . \frac{(-1)}{2} . \frac{(-x^2)^2}{2!} + \dots \right)$$

which is in the form of a Binomial expansion.

When you have determined it, you may care to substitute the closed form into the differential equation and verify that it is a solution.)

In this case the sum terminates if $\alpha = 2N+1$, leading to a polynomial solution of degree $2N+1$. These polynomials can be scaled so that $T_{2N+1}(1) = 1$ by choosing $a_1 = (-1)^N(2N+1)$.

The corresponding value of a_{2N+1} is given by

$$\begin{aligned} a_{2N+1} &= \frac{(-1)^N(2N+1)}{(2N+1)!} \prod_{r=0}^{N-1} ((2r+1)^2 - (2N+1)^2) \\ &= \frac{1}{(2N)!} \prod_{r=0}^{N-1} ((2N+1)^2 - (2r+1)^2) \\ &= \frac{1}{(2N)!} \prod_{r=0}^{N-1} (2N+2r+2)(2N-2r) \\ &= \frac{2^{2N}(1.2.\dots.N.(N+1).\dots(2N-1).(2N)}{(2N)!} \\ &= 2^{2N} \end{aligned}$$

The first three odd Chebychev polynomials are

$$\begin{aligned} T_1(x) &= x \\ T_3(x) &= 4x^3 - 3x \\ T_5(x) &= 16x^5 - 20x^3 + 5x \end{aligned}$$

Orthogonality of the solutions.

In order to determine the orthogonality properties of the Chebychev polynomials, it is necessary to rewrite the original equation in the self-adjoint form;

$$\frac{d}{dx}\left(r(x) \frac{dy}{dx} \right) = w(x)y$$

Expanding this form, we obtain

$$r(x)y'' + r'(x)y' - w(x)y = 0$$

which can be written as

$$y'' + \frac{r'(x)}{r(x)}y' - \frac{w(x)}{r(x)}y = 0$$

which corresponds to our original equation if

$$\frac{r'(x)}{r(x)} = \frac{x}{x^2-1} \;;\; -\frac{w(x)}{r(x)} = \frac{\alpha^2}{1-x^2}$$

Integrating the first of these equations we obtain

$$\log(r(x)) = \frac{1}{2}\log(1-x^2) \;;\; r(x) = \sqrt{1-x^2} \;;\; w(x) = -\frac{\alpha^2}{\sqrt{1-x^2}}$$

Hence the self-adjoint form of the equation is

$$\frac{d}{dx}\left(\sqrt{1-x^2}\frac{dy}{dx}\right) = -\frac{\alpha^2}{\sqrt{1-x^2}}y$$

which in the case of the n^{th} Chebychev polynomial gives

$$n^2\frac{T_n(x)}{\sqrt{1-x^2}} = -\frac{d}{dx}\left(\sqrt{1-x^2}\frac{dT_n(x)}{dx}\right) \;.$$

Therefore,

$$\begin{aligned} n^2\int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx &= -\int_{-1}^{1}\left[\frac{d}{dx}\left(\sqrt{1-x^2}\frac{dT_n(x)}{dx}\right)\right]T_m(x)\,dx \\ &= -\left(\sqrt{1-x^2}\frac{dT_n(x)}{dx}\right)T_m(x)\Big|_{-1}^{1} \\ &\quad + \int_{-1}^{1}\sqrt{1-x^2}\frac{dT_n(x)}{dx}\frac{dT_m(x)}{dx}\,dx \end{aligned}$$

Since $\sqrt{1-x^2}$ vanishes at $x=\pm 1$, and T_n' and T_m, being polynomials, are finite, the partial integral is zero, and

$$n^2\int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1}\sqrt{1-x^2}\frac{dT_n(x)}{dx}\frac{dT_m(x)}{dx}\,dx$$

Similarly,

$$m^2\int_{-1}^{1}\frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\,dx = \int_{-1}^{1}\sqrt{1-x^2}\frac{dT_m(x)}{dx}\frac{dT_n(x)}{dx}\,dx$$

and since the right hand sides are the same,

$$n^2\int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = m^2\int_{-1}^{1}\frac{T_m(x)T_n(x)}{\sqrt{1-x^2}}\,dx$$

so that, if $n \neq m$,

$$\int_{-1}^{1}\frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = 0 \;.$$

This is the orthogonality relationship for the Chebychev polynomials.

It differs from that for the Legendre polynomials by the inclusion of the factor $1/\sqrt{1-x^2}$ in the integrand. This factor is known as a *weight factor*. Using different weight factors gives rise to different families of orthogonal polynomials (and associated differential equations).

The Chebychev polynomials satisfy a three term recurrence relation of the form

$$xT_n(x) = c_{n+1}T_{n+1}(x) + c_{n-1}T_{n-1}(x) \ .$$

For $n \geq 1$, (which is implied by the expression, since $(n-1) \geq 0$) the leading coefficient in the expansion of $xT_n(x)$ is $2^{n-1}x^{n+1}$, while the leading term on the right hand side is $c_{n+1}2^n x^{n+1}$. Equating these terms, we see that $c_{n+1} = \frac{1}{2}$.

Now, substituting $x = 1$, we have

$$1.T_n(1) = 1 = c_{n+1}T_{n+1}(1) + c_{n-1}T_{n-1}(1) = c_{n+1} + c_{n-1} \ .$$

Therefore, $c_{n-1} = \frac{1}{2}$ also, and the recurrence relation for the Chebychev polynomials is

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x)$$

which you may care to verify for the values already given.

Although the power series solutions (1) and (2) which we have derived are valid for $|x| < 1$, they do not give us any information about the behaviour of the solutions of the differential equation as $x \to 1-$ and $x \to -1+$.

To investigate the behaviour of the solutions near $x = 1$, let us substitute $x = 1 - \epsilon t$.

Then

$$\frac{dy}{dx} = -\frac{1}{\epsilon}\frac{dy}{dt} \ ; \ \frac{d^2y}{dx^2} = \frac{1}{\epsilon^2}\frac{d^2y}{dt^2}$$

and the differential equation becomes

$$\begin{aligned}\frac{(1-(1-2\epsilon t+\epsilon^2t^2))}{\epsilon^2}\frac{d^2y}{dt^2} + \frac{1-\epsilon t}{\epsilon}\frac{dy}{dt} + \alpha^2 y = 0 \\ (2t - \epsilon t^2)\ddot{y} + (1-\epsilon t)\dot{y} + \epsilon\alpha^2 y = 0 \\ 2t\ddot{y} + \dot{y} \simeq 0\end{aligned}$$

This shows that $x = 1$ is a regular singular point of the differential equation.

The indicial equation associated with this Euler equation is $2r(r-1) + r = 0$, whose roots are $r = 0$ and $r = \frac{1}{2}$. These indices are independent of the value of α.

Therefore near $x = 1$, we could use the method of Frobenius to find one solution, y_3 say, which has a power series expansion

$$y_3 = \sum_{r=0}^{\infty} b_n(1-x)^n$$

and a second solution y_4 in the form

$$y_4 = \sqrt{1-x}\sum_{r=0}^{\infty} c_n(1-x)^n \ .$$

Since y_3 and y_4 are solutions of the differential equation for $-1 < x < 1$, there are constants such that

$$\begin{aligned}y_1 = A_1y_3 + B_1y_4 \\ y_2 = A_2y_3 + B_2y_4\end{aligned}$$

This means that the solutions y_1 and y_2 have finite limits (A_1 and A_2 respectively) as $x \to 1-$, but that in general y_1' and y_2' tend to infinity as $x \to 1-$. The exceptions to this second rule are precisely the cases when we have the Chebychev polynomial solutions.

In similar fashion we can show that $x = -1$ is also a regular singular point with indices $r = 0$ and $r = \frac{1}{2}$, so that there are particular solutions

$$y_5 = \sum_{r=0}^{\infty} b_n (1+x)^n$$

$$\text{and} \qquad y_6 = \sqrt{1+x} \sum_{r=0}^{\infty} c_n (1+x)^n$$

which describe the behaviour of the solutions in general near $x = -1$.

Considering the solutions y_4 and y_6 suggests that it might be interesting to see what happens if we make the substitution $y = \sqrt{1-x^2}\, z$ in the differential equation.

If $y = \sqrt{1-x^2}\, z$, then

$$y' = \sqrt{1-x^2}\, z' - \frac{x}{\sqrt{1-x^2}}\, z$$

$$y'' = \sqrt{1-x^2}\, z'' - \frac{2x}{\sqrt{1-x^2}}\, z' - \left(\frac{1}{\sqrt{1-x^2}} + \frac{x^2}{(\sqrt{1-x^2})^3} \right) z$$

$$(1-x^2)y'' - xy' + \alpha^2 y = \sqrt{1-x^2}\left((1-x^2)z'' - 3xz' + (\alpha^2 - 1)z \right) .$$

Our substitution therefore leads to a new differential equation

$$(1-x^2)z'' - 3xz' + (\alpha^2 - 1)z = 0$$

which is similar to the equation with which we began. In itself this is an encouraging sign, which suggests that we are on to something useful here.

Substituting $z = \sum_{n=0}^{\infty} a_n x^n$ into this equation gives the equation

$$\sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}x^n = \sum_{n=0}^{\infty} ((n-1)n + 3n - (\alpha^2 - 1))a_n x^n$$

and the recurrence relation

$$(n+1)(n+2)a_{n+2} = ((n+1)^2 - \alpha^2)a_n \ .$$

This relation leads to terminating (polynomial) expansions if $\alpha^2 = (n+1)^2$ for some integer $n \geq 0$. Therefore, apart from the case $\alpha = 0$, a second solution to our original equation when α is an integer n can be found in the form

$$y = \sqrt{1-x^2}U_{n-1}(x)$$

where $U_{n-1}(x)$ is a polynomial of degree $n-1$ in x.

For the first few values we can solve the recurrence relation in the terminating cases. (There is no point in finding the infinite power series solutions, since we know that they have the form $T_n(x)/\sqrt{1-x^2}$.)

If $\alpha = 1$,

$$2a_2 = 0.a_0 \; ; \; z_1 = a_0$$

If $\alpha = 2$,

$$6a_3 = 0.a_1 \; ; \; z_2 = a_1 x$$

If $\alpha = 3$,

$$2a_2 = (1-9)a_0 \; ; \; 12a_4 = 0.a_2 \; ; \; z_1 = a_0(1-4x^2)$$

If $\alpha = 4$,

$$6a_3 = (4-16)a_1 \; ; \; 20a_5 = 0.a_3 \; ; \; z_2 = a_1(x-2x^3)$$

If we scale these polynomials so that $U_{n-1}(1) = n$, then the first four are

$$\begin{aligned} U_0(x) &= 1 \\ U_1(x) &= 2x \\ U_2(x) &= 4x^2 - 1 \\ U_3(x) &= 8x^3 - 4x \end{aligned}$$

Note that with this scaling the coefficient of x^n in $U_n(x)$ is 2^n.

Students might like to show that these new polynomials (which are called Chebychev polynomials of the second kind) are orthogonal on the interval $[-1, 1]$ with weight factor $\sqrt{1-x^2}$.

This can be done either by rewriting the new differential equation in self-adjoint form, or by considering the orthogonality of the functions $\hat{U}_n = \sqrt{1-x^2}U_n$ with respect to the original equation.

The three term recurrence relation satisfied by these polynomials is

$$U_{n+1} = 2xU_n - U_{n-1}$$

which is precisely the same as that satisfied by the Chebychev polynomials of the first kind.

Both these families of polynomials are solutions of the second order linear **difference** equation

$$Y_{n+1} - 2xY_n + Y_{n-1} = 0 \; ,$$

and the general solution of this equation is

$$Y_n = AT_n(x) + BU_n(x) \; .$$

There is in fact a progression from $T_n(x)$ (the Chebychev polynomials of the first kind) to $P_n(x)$ (the Legendre polynomials) to $U_n(x)$.

The differential equations satisfied by these three classes of polynomial are

$(T_n(x))$ $$(1-x^2)y'' - xy' + n^2 y = 0$$

$(P_n(x))$ $$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

$(U_n(x))$ $$(1-x^2)y'' - 3xy' + n(n+2)y = 0$$

respectively, and the orthogonality relationships are

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}\,dx = 0 \quad n \neq m$$

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0 \quad n \neq m$$

$$\int_{-1}^{1} \sqrt{1-x^2}\,U_n(x)U_m(x)\,dx = 0 \quad n \neq m$$

To summarise, when α is a positive integer the general solution of Chebychev's differential equation on the interval $(-1,1)$ is

$$y = AT_n(x) + B\sqrt{1-x^2}U_{n-1}(x)\ .$$

At $x = 1$, we can choose the value of $y(1)$, which determines A. However, we cannot choose $y'(1)$ arbitrarily. Either we have $B = 0$, in which case $y'(1) = AT_n'(1) = An^2$,[1] or else $y'(1)$ is infinite. Hence there is no solution for arbitrary $y(1) = a, y'(1) = b$, which verifies the breakdown of the existence theorem.

Solutions for $|x| > 1$

In order to determine the solutions of the differential equation for $x > 1$ and $x < -1$, we make the substitution $t = 1/x$, and consider the solutions of this modified equation for $|t| < 1$.

If $t = 1/x$, then

$$\frac{d}{dx} = \frac{d}{dt}\frac{dt}{dx} = -\frac{1}{x^2}\frac{d}{dt} = -t^2\frac{d}{dt}\ .$$

Substituting into the differential equation gives

$$\begin{aligned}
\left(1-\frac{1}{t^2}\right)\left((-t^2)\frac{d}{dt}\left((-t^2)\frac{dy}{dt}\right)\right) - \frac{1}{t}(-t^2)\frac{dy}{dt} + \alpha^2 y = 0 \\
(-t^2+1)\left[-2t\frac{dy}{dt} - t^2\frac{d^2y}{dt^2}\right] + t\frac{dy}{dt} + \alpha^2 y = 0 \\
t^2(t^2-1)\frac{d^2y}{dt^2} + (2t^3-t)\frac{dy}{dt} + \alpha^2 y = 0 \\
t^2\frac{d^2y}{dt^2} + t\frac{2t^2-1}{t^2-1}\frac{dy}{dt} + \frac{\alpha^2}{t^2-1}y = 0
\end{aligned}$$

This shows that the equation has a regular singular point at $t = 0$, i.e. $x = \infty$. The Euler approximation near the singular point is

$$t^2\ddot{y} + t\dot{y} - \alpha^2 y = 0\ ,$$

for which the indicial equation is

$$r(r-1) + r - \alpha^2 = 0\ ;\ r^2 = \alpha^2\ ;\ r = \pm\alpha\ .$$

[1] Exercise: Derive this result for $T_n'(1)$ in one line! What are the corresponding results for $P_n'(1)$ and $U_n'(1)$?

In general therefore, we can use the method of Frobenius to determine two solutions in the forms

$$y_1 = |t|^\alpha \sum_{n=0}^{\infty} a_n t^n = |x|^{-\alpha} \sum_{n=0}^{\infty} a_n x^{-n}$$

and

$$y_2 = |t|^{-\alpha} \sum_{n=0}^{\infty} b_n t^n = |x|^{\alpha} \sum_{n=0}^{\infty} b_n x^{-n} \ .$$

When $\alpha = 0$, there is only one solution of this form, and we will need to find a second solution by other means.

We might also expect trouble when α is a positive integer N, but in these cases the Chebychev polynomial solution $T_N(x)$ behaves like x^N, and the second, potentially troublesome, solution y_2 reduces to this polynomial.

Assume for the moment that $\alpha > 0$. If $y = \sum_{n=0}^{\infty} a_n t^{\alpha+n}$,

$$y' = \sum_{n=0}^{\infty} (\alpha+n) a_n t^{\alpha+n-1} \ ; \ ty' = \sum_{n=0}^{\infty} (\alpha+n) a_n t^{\alpha+n}$$

$$t^3 y' = \sum_{n=0}^{\infty} (\alpha+n) a_n t^{\alpha+n+2} \ = \ \sum_{n=2}^{\infty} (\alpha+n-2) a_{n-2} t^{\alpha+n}$$

$$y'' = \sum_{n=0}^{\infty} (\alpha+n)(\alpha+n-1) a_n t^{\alpha+n-2} \ ; \ t^2 y'' = \sum_{n=0}^{\infty} (\alpha+n)(\alpha+n-1) a_n t^{\alpha+n}$$

$$t^4 y'' = \sum_{n=0}^{\infty} (\alpha+n)(\alpha+n-1) a_n t^{\alpha+n+2} \ = \ \sum_{n=2}^{\infty} (\alpha+n-2)(\alpha+n-3) a_{n-2} t^{\alpha+n}$$

Substituting these expansions into the differential equation we obtain

$$\sum_{n=2}^{\infty} (\alpha+n-2)(\alpha+n-3) a_{n-2} t^{\alpha+n} - \sum_{n=0}^{\infty} (\alpha+n)(\alpha+n-1) a_n t^{\alpha+n} +$$
$$+2 \sum_{n=2}^{\infty} (\alpha+n-2) a_{n-2} t^{\alpha+n} - \sum_{n=0}^{\infty} (\alpha+n) a_n t^{\alpha+n} + \alpha^2 \sum_{n=0}^{\infty} a_n t^{\alpha+n} = 0$$

Gathering together the corresponding powers of t, gives the relations

$$\left(-(\alpha)(\alpha-1) - \alpha + \alpha^2\right) a_0 = 0$$
$$\left(-(\alpha+1)(\alpha) - (\alpha+1) + \alpha^2\right) a_1 = 0$$
$$\left((\alpha+n-2)(\alpha+n-3) + 2(\alpha+n-2)\right) a_{n-2} = \left((\alpha+n)(\alpha+n-1) + (\alpha+n) - \alpha^2\right) a_n$$
$$\text{i.e.} \qquad (\alpha+n-2)(\alpha+n-1) a_{n-2} = n(n+2\alpha) a_n$$

Of these, the first is satisfied identically, confirming our choice of the index α, while the second gives $a_1 = 0$.

The last expression is the recurrence relation for determining the remaining coefficients.

Since it expresses a_n as a multiple of a_{n-2}, and since $a_1 = 0$, all the odd coefficients vanish, and the expansion is essentially in powers of t^2.

Setting $n = 2m$ in the recurrence relation, and rearranging the terms, gives

$$a_{2m} = \frac{(m + \frac{\alpha-2}{2})(m + \frac{\alpha-1}{2})}{m(m+\alpha)} a_{2(m-1)}$$

Considering the factors separately:

$$\left(m + \frac{\alpha-2}{2}\right)\Gamma\left(m + \frac{\alpha-2}{2}\right) = \Gamma\left(m + \frac{\alpha}{2}\right)$$
$$\left(m + \frac{\alpha-1}{2}\right)\Gamma\left(m + \frac{\alpha-1}{2}\right) = \Gamma\left(m + \frac{\alpha+1}{2}\right)$$
$$(m+\alpha)\Gamma(m+\alpha) = \Gamma(m+1+\alpha)$$
$$m((m-1)!) = m!$$

so that if

$$a_0 = \frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\alpha+1}{2})}{0!\Gamma(1+\alpha)}$$
$$a_{2m} = \frac{\Gamma(m+\frac{\alpha}{2})\Gamma(m+\frac{\alpha+1}{2})}{m!\Gamma(m+1+\alpha)}$$

and

$$y_1 = |t|^{\alpha} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{\alpha}{2})\Gamma(m+\frac{\alpha+1}{2})}{m!\Gamma(m+1+\alpha)} t^{2m}$$

is a solution for $0 < |t| < 1$.

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We obtain the second solution in general by replacing α by $-\alpha$. If α is not an integer,

$$y_2 = |t|^{-\alpha} \sum_{m=0}^{\infty} \frac{\Gamma(m-\frac{\alpha}{2})\Gamma(m-\frac{\alpha-1}{2})}{m!\Gamma(m+1-\alpha)} t^{2m} \ .$$

When $\alpha = N$, a positive integer,the recurrence relation for the coefficients is

$$4m(m-N)b_{2m} = ((2m-1)-N)((2m-2)-N)b_{2(m-1)}$$

which becomes indeterminate for b_{2m} when $m = N$.

However, one of the factors $((2m-1)-N)$ or $((2m-2)-N)$ vanishes for some $m \leq N$, so that we can choose $b_{2m} = 0$ for $m > [N/2]$. The solution then has a finite expansion, which, as previously indicated, is the N^{th} Chebychev polynomial.

When $\alpha = 0$, it is most convenient to solve the original equation from scratch. In self adjoint form it becomes

$$\frac{d}{dx}\left(\sqrt{x^2-1}\frac{dy}{dx}\right) = 0 \ ,$$

(where in this case $\log|x^2-1| = \log(x^2-1)$) so that

$$\sqrt{x^2-1}\frac{dy}{dx} = c_1$$
$$\frac{dy}{dx} = \frac{c_1}{\sqrt{x^2-1}}$$
$$y = c_2 + c_1 \text{arcosh}\,(|x|) = c_2 + c_1\log(|x| + \sqrt{x^2-1})$$

If $x = 1/t$,

$$\begin{aligned}\log(|x| + \sqrt{x^2-1}) &= \log(|x|) + \log(1+\sqrt{1-x^{-2}})\\ &= -\log(|t|) + \log(1+\sqrt{1-t^2})\end{aligned}$$

which is the usual form for the second solution when the indices are equal.

When α is a positive integer, we can also modify the solutions derived for $|x| < 1$ to suit the case $|x| > 1$.

The polynomials $T_n(x)$ obviously remain as solutions for $|x| > 1$.

For the other solutions, we need to alter the factor $\sqrt{1-x^2}$, which is the form suitable for $|x| < 1$, to become $\sqrt{x^2-1}$.

Hence, for $|x| > 1$, the general solution of the original equation can be written as

$$y = AT_n(x) + B\sqrt{x^2-1}U_{n-1}(x)\ .$$

The leading terms in the expansions of these two particular solutions are

$$\begin{aligned}T_n(x) &= 2^{n-1}x^n + \ldots\\ \sqrt{x^2-1}U_{n-1}(x) &= xU_{n-1}(x) + \ldots\\ &= 2^{n-1}x^n + \ldots\end{aligned}$$

Therefore, $\sqrt{x^2-1}U_{n-1}(x) - T_n(x)$ is a solution of the equation which is smaller than x^n near $x = \infty$, or equivalently, bigger than t^{-n} near $t = 0$.

Because such a solution must be a linear combination of the two solutions we have derived by Frobenius' method about $t = 0$, it follows that this difference must in fact be a multiple of the solution y_1.

i.e.

$$\sqrt{x^2-1}U_{n-1}(x) = T_n(x) + c|x|^{-n}\sum_{m=0}^{\infty} a_{2m}x^{-2m}\ .$$

This condition is very restrictive.

It effectively defines both kinds of Chebychev polynomial to within a multiplicative constant.

For instance, suppose we want to determine U_2 and T_3.

The most general possible form for U_2 is $ax^2 + bx + c$.

Multiplying by $\sqrt{x^2-1}$, and expanding appropriately, we have

$$\begin{aligned}(ax^2+bx+c)\sqrt{x^2-1} &= (ax^3+bx^2+cx)\sqrt{1-x^{-2}}\\ &= (ax^3+bx^2+cx)(1-\frac{1}{2}x^{-2}-\frac{1}{8}x^{-4}-\frac{1}{16}x^{-6}-\dots)\\ &= ax^3+bx^2+\left(c-\frac{1}{2}a\right)x-\frac{b}{2}\\ &\quad -\left(\frac{c}{2}+\frac{a}{8}\right)x^{-1}-\frac{b}{8}x^{-2}\\ &\quad -\left(\frac{c}{8}+\frac{a}{16}\right)x^{-3}+\dots\end{aligned}$$

The conditions that the expansion does not contain x^{-1} and x^{-2} terms give $a=-4c$ and $b=0$, so that

$$U_2(x)=-c(4x^2-1)\ ;\ T_3(x)=-c(4x^3-3x) \text{ for some } c\ .$$

Finally, if we set $R_n(x)=T_n(x)-\sqrt{x^2-1}U_{n-1}(x)$, we have

$$R_1 = x-\sqrt{x^2-1} = 1/(x+\sqrt{x^2-1})$$
$$R_1^2 = x^2-2x\sqrt{x^2-1}+x^2-1 = 2x^2-1-\sqrt{x^2-1}(2x) = R_2$$

For $n>2$, we can determine R_n by using the recurrence relation for the Chebychev polynomials; $R_{n+1}=2xR_n-R_{n-1}$.

Assuming inductively that $R_k=(x-\sqrt{x^2-1})^k$ for $k\le n$, we have

$$\begin{aligned}R_{n+1} &= (x-\sqrt{x^2-1})^{n-1}\left(2x(x-\sqrt{x^2-1})-1\right)\\ &= (x-\sqrt{x^2-1})^{n-1}\left(2x^1-1-2x\sqrt{x^2-1}\right)\\ &= (x-\sqrt{x^2-1})^{n+1}\end{aligned}$$

so that this formula is correct for all $n\ge 1$, and we have found (to within a multiplicative constant) a closed form for the solution y_1.

Summary

The equation which we have been studying belongs to a special class of differential equations which have the property that they have three regular singular points.

Legendre's differential equation and the Hypergeometric equation are other important members of this class. Indeed, any equation of this type can be transformed into a hypergeometric equation by a suitable change of variable – $X=(x+1)/2$ for the cases of Legendre's and Chebychev's equations. Equations of this class also have important self-similarity properties about which you can read in, for example, *Special Functions of Mathematical Physics and Chemistry* by I.N. Sneddon or *Higher Transcendental Functions* by Erdelyi, Magnus, Oberhettinger and Tricomi.

Hopefully, these notes will indicate the wide range of properties which can be derived from just one differential equation.