**MP213**

LAPLACE TRANSFORMS

SELF-HELP EXERCISES USING MAPLE II

Discontinuous functions

**Maple** has the ability to handle discontinuous functions using the *Heaviside* function.

For instance, the function

$$f = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t \end{cases}$$

can be represented by

**f:= 1 - Heaviside(t-1);**

which you can check by typing

**plot(f,t=0..2);**.

*You will need to close the 'PLOT' window in order to return to the main Maple routine.*

We can also solve the equation

$$\ddot{y} + y = f \ , \quad y(0) = 0 \ ; \ \dot{y}(0) = 0$$

by setting

**eq:= diff(y(t),t,t) + y(t) = f ;**

and proceeding as before –

**leq:= laplace(eq,t,s);**

**sub:= subs(y(0)=0,D(y)(0)= 0,leq);**

**Y:= solve(sub,laplace(y(t),t,s));**

**Y1:= normal(Y);**

Since the expression is not a rational function, we skip the next step in the previous approach.

**yp:= invlaplace(Y1,s,t);**

You should plot this solution using

**plot(yp,t=0..10);**

*Remember to close the 'PLOT' window.*

You will have noticed that the solution is continuous in spite of the discontinuous forcing function. The discontinuity occurs in the second derivative.

Similarly, the function

$$f_1 = \begin{cases} e^t & 0 < t < 1 \\ e^{-t} & 1 < t < 2 \\ 0 & 2 < t \end{cases}$$

which has been considered in the lecture notes can be represented by
**f1:= exp(t)*(1 - Heaviside(t-1)) + exp(-t)*(Heaviside(t-1) - Heaviside(t-2));**

The graph of $f_1$ can be seen by typing
**plot(f1,t=0..3);**,
and you can verify its Laplace transform by typing
**laplace(f1,t,s);**.

**Exercises.**
1. Find the solution of the equation

$$\ddot{y} - y = f_1 \ , \quad y(0) = 0 \ ; \ \dot{y}(0) = 0$$

where $f_1$ is the function above.

The equation is entered as
**eq:= diff(y(t),t,t) - y(t) = f;**

Then
**leq:= laplace(eq,t,s);**

**sub:= subs(y(0)=0,D(y)(0)=0,leq);**

**Y:= solve(sub,laplace(y(t),t,s));**

**Y1:= normal(Y);**
(If you omit this step, maple gives the final answer as a convolution!)

**yp:= invlaplace(Y1,s,t);**

2. Let $f_a$ be the function given by

$$f_a = \begin{cases} t/a^2 & 0 < t < a \\ (2a-t)/a^2 & a < t < 2a \\ 0 & 2a < t \end{cases} .$$

(a) Express $f_a$ in terms of the Heaviside function.

```
fa:= (1 - Heaviside(t-a))*t/a^ 2
     + Heaviside(t-a)*(2*a-t)/a^ 2
     -Heaviside(t-2*a)*(2*a-t)/a^ 2;
```

(The second term needs to be spread out, since the Laplace transform program seems unable to cope with two Heaviside functions at once)

(b) Use the Laplace transform to find the solution of the equation

$$\ddot{y} + y = f_a \ , \quad y(0) = 0 \ , \ \dot{y}(0) = 0$$

for the cases $a = 1/2$, $a = 1/10$ and $a = 1/100$.

Plot the solutions for $0 \leq t \leq 10$.

What do you think will be the limiting solution as $a \to 0+$?

Firstly, we evaluate $f_a$.

```
f:= subs(a = 1/2,fa):
```

(The substitution has to be made at this point, since, for some reason, **maple** cannot handle a parameter in the Heaviside function.)

then we set up the differential equation

```
eq:= diff(y(t),t,t) + y(t) = f;
```

and proceed as usual

```
leq:= laplace(eq,t,s);
```

```
sub:= subs(y(0)=0,D(y)(0)=0,leq);
```

```
Y:= solve(sub,laplace(y(t),t,s)):
```

```
Y1:= normal(Y);
```

```
yp:= invlaplace(Y1,s,t):
```

To plot the solution for $0 \le t \le 10$, type
**plot(yp,t=0..10);**

To repeat the calculations for $a = 1/10$ and $a = 1/100$, you do not need to reenter all the instructions.

Instead, use the up-arrow on the right of the keyboard to scroll up to the command
**f:= subs(a=1/2,fa);**
then use the left/right cursors to move the vertical bar | until you have
**f:= subs(a=1/2|,fa);**
press the back-space key to delete **2**, and then type**10**. This should give you
**f:= subs(a=1/10,fa);**

If you now press the return key, the program will execute that command and jump to the next command in the list, i.e.
**eq:= diff(y(t),t,t) + y(t) = f;**
Continuing in this way, you can repeat all the steps in the solution procedure.

3. Repeat exercise 2 using

$$f_a = \begin{cases} a^{-2} & 0 < t < a \\ -a^{-2} & a < t < 2a \\ 0 & 2a < t \end{cases} .$$

Write
**fa:= (1 - 2*Heaviside(t-a) + Heaviside(t-2*a))/a^ 2;**
and then use the cursor keys to repeat the instructions above.