BOUNDARY VALUE PROBLEMS AND PARTIAL DIFFERENTIAL EQUATIONS

While there is no difficulty finding a unique solution to the initial value problem

$$y'' + y = 0 \; ; \; y(x_0) = a \; , \; y'(x_0) = b \; ,$$

the situation changes dramatically if we specify conditions at two distinct values of x.

For example;

$$y'' + y = 0 \; ; \; y(0) = 0 \; , \; y(1) = 1 \tag{1}$$

has a unique solution $y = \sin x / \sin 1$,

$$y'' + y = 0 \; ; \; y(0) = 0 \; , \; y(\pi) = 1 \tag{2}$$

has no solution, and

$$y'' + y = 0 \; ; \; y(0) = 0 \; , \; y(\pi) = 0 \tag{3}$$

has an infinite set of solutions $y = c \sin x$.

Eigenvalues and Eigenfunctions.

Consider the slightly more general differential equation

$$y'' + \lambda y = 0 \; .$$

If we impose *homogeneous boundary conditions* of the form

$$\begin{aligned} k_{11} y(a) + k_{12} y'(a) &= 0 \qquad (k_{11}, k_{12}) \neq (0,0) \\ k_{21} y(b) + k_{22} y'(b) &= 0 \qquad (k_{21}, k_{22}) \neq (0,0) \end{aligned}$$

we can ensure that there will always be at least one solution to the boundary value problem, namely $y \equiv 0$. We can now ask the question whether there are values of λ for which an infinite set of solutions exist. Such values of λ are called *eigenvalues* and the corresponding solutions are called *eigenfunctions*.

e.g. 1.

Consider the problem

$$y'' + \lambda y = 0 \; ; \; y(0) = 0 \; , \; y(1) = 0 \; .$$

If $\lambda > 0$, set $\lambda = \omega^2$. Then the general solution of the equation

$$y'' + \omega^2 y = 0$$

is

$$y = a \cos(\omega x) + b \sin(\omega x) \; .$$

The condition $y(0) = 0$ implies $a = 0$, and now the second condition gives

$$b \sin(\omega) = 0 \; .$$

As well as the solution $b = 0$ which leads to $y \equiv 0$, we have the possibility $\sin(\omega) = 0$, which has the solutions $\omega = n\pi$. Therefore, if $\lambda = n^2\pi^2$, $n = 1, 2, \ldots$,we have the non-trivial solution sets $y = b\sin(n\pi x)$.

If $\lambda < 0$, set $\lambda = -p^2$. Then the general solution of the equation

$$y'' - p^2 y = 0$$

can be written

$$y = a\cosh(px) + b\sinh(px)\ .$$

The condition $y(0) = 0$ implies $a = 0$ again, and the second condition gives

$$b\sinh(p) = 0\ .$$

However, the function $\sinh x$ vanishes only for $x = 0$, and since $p \neq 0$ we must take $b = 0$ and hence $y \equiv 0$.

Finally, if $\lambda = 0$, the general solution of

$$y'' = 0$$

is

$$y = a + bx\ ,$$

and the boundary conditions ensure that $a = b = 0$ and $y \equiv 0$.

Therefore the eigenvalues for this problem are $\lambda = n^2\pi^2$ and the corresponding eigenfunctions are $y(x) = \sin(n\pi x)$.

e.g. 2.

Consider the boundary value problem

$$y'' + \lambda y = 0\ ; y'(0) = 0\ ,\ y'(1) = 0\ .$$

For $\lambda < 0$ set $\lambda = -p^2$. Then the general solution of

$$y'' - p^2 y = 0$$

can be written in the form

$$y = a\cosh(px) + b\sinh(px)\ ;\ y' = pa\sinh(px) + pb\cosh(px)$$

The condition $y'(0) = 0$ gives $b = 0$, and the condition $y'(1) = 0$ gives

$$pa\sinh(p) = 0\ ;\ a = 0 \quad \text{since} \quad p \neq 0\ .$$

For $\lambda = 0$, the general solution is

$$y = a + bx\ ;\ y' = b$$

so that the conditions give $b = 0$, and $y = a$ is an infinite solution set.

For $\lambda > 0$ set $\lambda = \omega^2$. The general solution of

$$y'' + \omega^2 y = 0$$

is

$$y = a\cos(\omega x) + b\sin(\omega x)\ ;\ y' = -a\omega\sin(\omega x) + b\omega\cos(\omega x)\ .$$

The condition $y'(0) = 0$ gives $b = 0$, and the condition $y'(1) = 0$ gives

$$a\omega\sin(\omega) = 0$$

so that, as well as the solution $a = 0$ we have $\omega = n\pi$, leading to the eigenvalues $\lambda = n^2\pi^2$ and the corresponding eigenfunctions $y = \cos(n\pi x)$.

e.g. 3.

Consider the problem

$$y'' + \lambda y = 0 \ ; \ y(0) = 0 \ , \ y'(1) = 2y(1) \ .$$

For $\lambda < 0$, set $\lambda = -p^2$. The general solution of

$$y'' - p^2 y = 0$$

is

$$y = a\cosh(px) + b\sinh(px) \ ,$$

and the condition $y(0) = 0$ gives $a = 0$. The second condition gives

$$\begin{aligned} pb\cosh(p) &= 2b\sinh(p) \\ \tanh(p) &= \frac{p}{2} \quad \text{if } b \neq 0 \\ p &\simeq 1.915 \end{aligned}$$

Therefore one eigenvalue is $-(1.915)^2$, and the corresponding eigenfunction is $y = \sinh(1.915x)$.

For $\lambda = 0$, the general solution is $y = a + bx$. The condition $y(0) = 0$ gives $a = 0$, and the condition $y'(1) = 2y(1)$ gives $b = 2b$, $b = 0$, so that the only solution is $y \equiv 0$.

Finally, for $\lambda > 0$, we set $\lambda = \omega^2$, so that the general solution is

$$y = a\cos(\omega x) + b\sin(\omega x) \ .$$

The condition $y(0) = 0$ gives $a = 0$, and the condition $y'(1) = 2y(1)$ gives

$$\begin{aligned} b\omega\cos(\omega) &= 2b\sin(\omega) \\ \tan(\omega) &= \frac{\omega}{2} \quad \text{if } b \neq 0 \end{aligned}$$

This equation has a sequence of solutions

$$\omega_n \simeq \left(n + \frac{1}{2}\right)\pi - \frac{4}{(2n+1)\pi} \qquad n = 1, 2, \ldots$$

so that the eigenvalues are $\lambda = \omega_n^2$, and the corresponding eigenfunctions are $y = \sin(\omega_n x)$.

A common feature of these examples, which is characteristic of all such eigenvalue problems on finite intervals, is that the eigenvalues are discrete and bounded below.

The other important property of the eigenfunctions is their orthogonality.

Suppose that we have the boundary value problem

$$y'' + \lambda y = 0 \ ; \ k_{11}y(a) + k_{12}y'(a) = 0 \ , \ k_{21}y(b) + k_{22}y'(b) = 0$$

If y_m is an eigenfunction corresponding to the eigenvalue λ_m, and y_n is an eigenfunction corresponding to the eigenvalue λ_n, where $\lambda_m \neq \lambda_n$, then

$$\begin{aligned}
\lambda_m \int_a^b y_m y_n \, dx &= -\int_a^b y_m'' y_n \, dx \\
&= -y_m' y_n \big|_a^b + \int_a^b y_m' y_n' \, dx \\
&= -\, y_m'(b) y_n(b) + y_m'(a) y_n(a) + \int_a^b y_m' y_n' \, dx \\
\lambda_n \int_a^b y_m y_n \, dx &= -\, y_n'(b) y_m(b) + y_n'(a) y_m(a) + \int_a^b y_m' y_n' \, dx \\
(\lambda_m - \lambda_n) \int_a^b y_m y_n \, dx &= \begin{vmatrix} y_m(b) & y_m'(b) \\ y_n(b) & y_n'(b) \end{vmatrix} - \begin{vmatrix} y_m(a) & y_m'(a) \\ y_n(a) & y_n'(a) \end{vmatrix} \; = \; 0 \; .
\end{aligned}$$

Hence the eigenfunctions are orthogonal on the interval $[a, b]$.

This means that, for finite sums at least, if

$$f(x) = \sum_{k=1}^{N} \alpha_k y_k(x)$$

then

$$\begin{aligned}
\int_a^b f(x) y_n(x) \, dx &= \sum_{k=1}^{N} \alpha_k \int_a^b y_k(x) y_n(x) \, dx \\
&= \alpha_n \int_a^b y_n^2(x) \, dx \\
\alpha_n &= \int_a^b f(x) y_n(x) \, dx \bigg/ \int_a^b y_n^2(x) \, dx
\end{aligned}$$

and we assume that this result extends to infinite sums also.

The method of separation of variables.

Consider the following partial differential equation (*the wave equation*).

$$\begin{gathered}\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \\ u(0,t) = 0 \ ; \ u(1,t) = 0 \\ u(x,0) = f(x) \ ; \ u_t(x,0) = g(x)\end{gathered}$$

The equation is linear, and we have homogeneous boundary conditions for fixed values of the variable x. Therefore we can attempt to solve the equation by using the method of separation of variables.

This method consists firstly of trying to find non-trivial solutions of the partial differential equation of the form $u = X(x)T(t)$ which satisfy the boundary conditions in x without worrying about the initial conditions in t. Substituting this form into the partial differential equation, we have

$$\begin{aligned}X''(x)T(t) &= \frac{1}{c^2}X(x)\ddot{T}(t) \\ \frac{X''(x)}{X(x)} &= \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)}\end{aligned}$$

since we are assuming that $X(x)$ and $T(t)$ are not identically 0;

$$\frac{X''(x)}{X(x)} = -\lambda = \frac{1}{c^2}\frac{\ddot{T}(t)}{T(t)}$$

since the left hand side is independent of t and the right hand side is independent of x;

$$X''(x) + \lambda X(x) = 0 \ ; \ X(0) = 0 \ , \ X(1) = 0 \ .$$

As we have seen above, the only non-trivial solutions of this problem correspond to $\lambda = n^2\pi^2$, $X(x) = \sin(n\pi x)$.

For these values of λ, $T(t)$ satisfies

$$\begin{gathered}\ddot{T}(t) + c^2n^2\pi^2T(t) = 0 \\ T(t) = a_n\cos(cn\pi t) + b_n\sin(cn\pi t)\end{gathered}$$

where a_n and b_n are arbitrary constants.

Therefore the functions $(a_n\cos(cn\pi t) + b_n\sin(cn\pi t))\sin(n\pi x)$ are solutions of

$$\begin{gathered}\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \\ u(0,t) = 0 \ , \ u(1,t) = 0\end{gathered}$$

for all choices of a_n and b_n, and for all $n = 1, 2, \ldots$, and because the equation is linear, this is also true for arbitrary linear combinations of these solutions. By extension we assume that the infinite sum

$$u(x,t) = \sum_{n=1}^{\infty}(a_n\cos(cn\pi t) + b_n\sin(cn\pi t))\sin(n\pi x)$$

is also a solution of this problem, and we will be able to solve the original problem provided we can choose the constants a_n and b_n so that $u(x,0) = f(x)$ and $u_t(x,0) = g(x)$. Substituting into the expression for $u(x,t)$ this implies

$$\sum_{n=1}^{\infty} a_n \sin(n\pi x) = f(x) \ ,$$

and

$$\sum_{n=1}^{\infty} cn\pi b_n \sin(n\pi x) = g(x) \ .$$

Now using the orthogonality of the eigenvalues we have

$$\begin{aligned} a_n &= \int_0^1 f(x)\sin(n\pi x)\,dx \Big/ \int_0^1 \sin^2(n\pi x)\,dx \\ &= 2\int_0^1 f(x)\sin(n\pi x)\,dx \\ b_n &= \frac{2}{cn\pi}\int_0^1 g(x)\sin(n\pi x)\,dx \ . \end{aligned}$$

For the purposes of this subject, we will assume that all these procedures are valid, but students should note that in order to justify this solution we need to ensure the convergence of the infinite sums which occur, prove that these sums are in fact twice differentiable with respect to both of the variables, and prove that it is possible to interchange the operations of differentiation and infinite summation.

As a particular example, let us take $f(x) = x(1-x)$, and $g(x) = 0$. The b_n are all 0, while the a_n are given by

$$\begin{aligned} a_n &= 2\int_0^1 x(1-x)\sin(n\pi x)\,dx \\ &= -\frac{2}{n\pi}\cos(n\pi x)x(1-x)\Big|_0^1 + \frac{2}{n\pi}\int_0^1 \cos(n\pi x)(1-2x)\,dx \\ &= \frac{2}{n^2\pi^2}\sin(n\pi x)(1-2x)\Big|_0^1 + \frac{4}{n^2\pi^2}\int_0^1 \sin(n\pi x)\,dx \\ &= -\frac{4}{n^3\pi^3}\cos(n\pi x)\Big|_0^1 \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{8}{n^3\pi^3} & \text{if } n \text{ is odd.} \end{cases} \end{aligned}$$

The solution for $u(x,t)$ is

$$u(x,t) = \frac{8}{\pi^3}\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3}\cos(c(2k+1)\pi t)\sin((2k+1)\pi x) \ .$$

Consider the following partial differential equation (*Laplace's Equation*).

$$\begin{gathered}\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0\\ u(0,y)=0\ ,\ u(\pi,y)=\frac{\pi}{2}-\left|\frac{\pi}{2}-y\right|\\ u(x,0)=0\ ,\ u(x,\pi)=x^2(\pi-x)\ .\end{gathered}$$

While the boundary data are given along the lines $x = const$ and $y = const$, in neither case are the boundary conditions homogeneous, so that the method of separation of variables cannot be applied directly.

However, the differential equation is linear, so that we can determine the required solution as the sum of the solutions of the simpler problems

$$\begin{gathered}\frac{\partial^2 u_1}{\partial x^2}+\frac{\partial^2 u_1}{\partial y^2}=0\\ u_1(0,y)=0\ ,\ u_1(\pi,y)=\frac{\pi}{2}-\left|\frac{\pi}{2}-y\right|\\ u_1(x,0)=0\ ,\ u_1(x,\pi)=0\ ,\end{gathered}$$

and

$$\begin{gathered}\frac{\partial^2 u_2}{\partial x^2}+\frac{\partial^2 u_2}{\partial y^2}=0\\ u_2(0,y)=0\ ,\ u_2(\pi,y)=0\\ u_2(x,0)=0\ ,\ u_2(x,\pi)=x^2(\pi-x)\ .\end{gathered}$$

The first of these has homogeneous boundary data along the lines $y = 0$ and $y = \pi$, while the second has homogeneous boundary data along the lines $x = 0$ and $x = \pi$.

In solving the first part, we look for solutions $u_1 = X(x)Y(y)$, where $Y(0) = Y(\pi) = 0$. Making the substitution in the equation, we have

$$\begin{gathered}X''(x)Y(y)+X(x)Y''(y)=0\\ \frac{X''(x)}{X(x)}=-\frac{Y''(y)}{Y(y)}=\lambda\\ Y''(y)+\lambda Y(y)=0\ ;\ Y(0)=0\ ,\ Y(\pi)=0\\ \lambda=n^2\quad n=1,2,\ldots\qquad Y(y)=\sin(ny)\end{gathered}$$

If $\lambda = n^2$, then the functions $X(x)$ satisfy

$$\begin{gathered}X''(x)-n^2X(x)=0\\ X(x)=a_n e^{nx}+b_n e^{-nx}\ ,\end{gathered}$$

and the solution u_1 can be expressed as

$$u_1(x,y) = \sum_{n=1}^{\infty} \left(a_n e^{nx} + b_n e^{-nx}\right) \sin(ny) \ .$$

In order to satisfy the other boundary conditions, we need

$$\sum_{n=1}^{\infty} (a_n + b_n) \sin(ny) = 0$$

and

$$\sum_{n=1}^{\infty} \left(a_n e^{n\pi} + b_n e^{-n\pi}\right) \sin(ny) = \frac{\pi}{2} - \left|\frac{\pi}{2} - y\right| \ .$$

The first of these equations gives $b_n = -a_n$, while the second gives

$$\begin{aligned}
2a_n \sinh(n\pi) &= \frac{2}{\pi}\left(\int_0^{\pi/2} y \sin(ny)\, dy + \int_{\pi/2}^{\pi} (\pi - y)\sin(ny)\, dy\right) \\
&= \frac{2}{\pi}\left(-\frac{1}{n}\cos(ny)y\Big|_0^{\pi/2} + \frac{1}{n}\int_0^{\pi/2} \cos(ny)\, dy\right. \\
&\qquad \left. -\frac{1}{n}\cos(ny)(\pi - y)\Big|_{\pi/2}^{\pi} - \frac{1}{n}\int_{\pi/2}^{\pi} \cos(ny)\, dy\right) \\
&= \frac{2}{\pi}\left(-\frac{\pi}{2n}\cos(n\pi/2) + \frac{1}{n^2}(\sin(n\pi/2) - \sin(0))\right. \\
&\qquad \left. + \frac{\pi}{2n}\cos(n\pi/2) - \frac{1}{n^2}(\sin(n\pi) - \sin(n\pi/2))\right) \\
&= \frac{4}{\pi n^2}\sin(n\pi/2) \\
&= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4}{\pi n^2} & \text{if } n \ = \ 4m+1 \\ -\frac{4}{\pi n^2} & \text{if } n \ = \ 4m+3 \end{cases}
\end{aligned}$$

Hence,

$$u_1(x,y) = \frac{4}{\pi}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} \frac{\sinh((2k+1)x)}{\sinh((2k+1)\pi)} \sin((2k+1)y) \ .$$

In identical fashion, we have

$$u_2(x,y) = \sum_{n=1}^{\infty} a_n \sinh(ny)\sin(nx)$$

where

$$\begin{aligned}
a_n \sinh(n\pi) &= \frac{2}{\pi}\int_0^\pi x^2(\pi - x)\sin(nx)\,dx \\
&= -\frac{2}{n\pi}\cos(nx)x^2(\pi - x)\Big|_0^\pi + \frac{2}{n\pi}\int_0^\pi (2\pi x - 3x^2)\cos(nx)\,dx \\
&= \frac{2}{n^2\pi}\sin(nx)(2\pi x - 3x^2)\Big|_0^\pi - \frac{2}{n^2\pi}\int_0^\pi (2\pi - 6x)\sin(nx)\,dx \\
&= -\frac{2}{n^3\pi}\cos(nx)(2\pi - 6x)\Big|_0^\pi - \frac{12}{n^3\pi}\int_0^\pi \cos(nx)\,dx \\
&= \frac{8}{n^3}\cos(n\pi) + \frac{4}{n^3} - \frac{12}{n^4\pi}\left(\sin(n\pi) - 0\right) \\
&= \frac{4}{n^3}\left(1 + 2(-1)^n\right)
\end{aligned}$$

Hence,

$$u_2(x,y) = 4\sum_{n=1}^{\infty} \frac{1 + 2(-1)^n}{n^3}\frac{\sinh(ny)}{\sinh(n\pi)}\sin(nx)\ .$$

The solution of the original problem is $u_1(x,y) + u_2(x,y)$.

Variations on a theme.

Consider the boundary value problem

$$\begin{gathered}
\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} \\
u(0,t) = 0 \quad ; \quad u_x(1,t) = 0 \\
u(x,0) = 1
\end{gathered}$$

We look first for solutions of the form $u(x,t) = X(x)T(t)$ where $X(0) = 0$ and $X'(1) = 0$. Substituting this form into the equation, we obtain

$$\begin{gathered}
X(x)\dot{T}(t) = X''(x)T(t) + X'(x)T(t) \\
\frac{\dot{T}(t)}{T(t)} = \frac{X''(x) + X'(x)}{X(x)} = -\lambda
\end{gathered}$$

so that the eigenfunctions satisfy the boundary value problem

$$X'' + X' + \lambda X = 0 \quad ; \quad X(0) = 0\ ,\ X'(1) = 0\ .$$

This is a linear differential equation with constant coefficients, for which the characteristic equation is

$$\begin{aligned}
m^2 + m + \lambda &= 0 \\
m &= \frac{-1 \pm \sqrt{1 - 4\lambda}}{2}
\end{aligned}$$

There are three cases to consider:

Case 1: $1-4\lambda > 0$. Set $\lambda = \frac{1}{4} - p^2$ $(p > 0)$, so that $m = -\frac{1}{2} \pm p$. The general solution of the equation is

$$X = a\exp\left(\left(-\frac{1}{2} - p\right)x\right) + b\exp\left(\left(-\frac{1}{2} + p\right)x\right)$$
$$X' = -\left(p + \frac{1}{2}\right)a\exp\left(\left(-\frac{1}{2} - p\right)x\right) + \left(p - \frac{1}{2}\right)b\exp\left(\left(-\frac{1}{2} + p\right)x\right)$$

If $X(0) = 0$, $a + b = 0$, $b = -a$.

If $X'(1) = 0$,

$$b\left(p + \frac{1}{2}\right)\exp\left(-\frac{1}{2} - p\right) + b\left(p - \frac{1}{2}\right)\exp\left(p - \frac{1}{2}\right) = 0$$
$$\text{either} \quad b = 0\ , \quad \text{or} \quad e^{-2p} = \frac{1-2p}{1+2p}\ ,$$

but this last equation is satisfied only by $p = 0$, so that there are no eigenvalues or eigenfunctions in this case.

Case 2: $1 - 4\lambda = 0$, so that $m = -\frac{1}{2}, -\frac{1}{2}$. The general solution of the equation is

$$X(x) = (a + bx)\exp\left(-\frac{x}{2}\right) \ ; \ X'(x) = \left(b - \frac{a}{2} - \frac{b}{2}x\right)\exp\left(-\frac{x}{2}\right) \ .$$

If $X(0) = 0$, then $a = 0$, and if $X'(1) = 0$, $(b/2)\exp(-1/2) = 0$, $b = 0$, so that $\lambda = 1/4$ is not an eigenvalue.

Case 3: $1 - 4\lambda < 0$. Set $\lambda = \frac{1}{4} + \omega^2$, so that $m = -\frac{1}{2} \pm i\omega$. The general solution is

$$X(x) = ae^{-x/2}\cos(\omega x) + be^{-x/2}\sin(\omega x) \ .$$

If $X(0) = 0$, then $a = 0$.

$$X'(x) = b\left(-\frac{1}{2}e^{-x/2}\sin(\omega x) + \omega e^{-x/2}\cos(\omega x)\right)$$
$$X'(1) = be^{-1/2}\left(-\frac{1}{2}\sin(\omega) + \omega\cos(\omega)\right)$$
$$X'(1) = 0 \text{ if } b = 0 \text{ or } \tan(\omega) = 2\omega$$

This gives the eigenvalues

$$\omega_n \simeq \left(n + \frac{1}{2}\right)\pi - \frac{1}{(2n+1)\pi} \qquad n = 0, 1, 2, \ldots$$
$$\lambda_n = \frac{1}{4} + \omega_n^2$$

with the corresponding eigenfunctions

$$X_n(x) = e^{-x/2}\sin(\omega_n x) \ .$$

The function $T(t)$ satisfies

$$\dot{T}(t) = -\lambda_n T(t) \qquad T(t) = a_n \exp(-\lambda_n t) \ .$$

Combining these fundamental solutions, we have a solution $u(x,t)$ of the form

$$u(x,t) = \sum_{n=0}^{\infty} a_n e^{-\lambda_n t} e^{-x/2} \sin(\omega_n x)$$

where we need to choose the coefficients a_n so that

$$\sum_{n=0}^{\infty} a_n e^{-x/2} \sin(\omega_n x) = 1 \ .$$

Unfortunately, $\int_0^1 X_m(x) X_n(x)\, dx \neq 0$ for $m \neq n$, so that the technique which we have used previously does not apply.

The adjoint operator.

Let $\mathbb{L}$ be the linear differential operator defined on $[\alpha, \beta]$ by

$$\mathbb{L}(y) = a_2(x) y''(x) + a_1(x) y'(x) + a_0(x) y(x) \ .$$

For arbitrary $y, v \in C^2[\alpha, \beta]$,

$$\begin{aligned}
\int_\alpha^\beta \mathbb{L}(y) v\, dx = & \int_\alpha^\beta \left[y'' a_2 v + y' a_1 v + y a_0 v \right] dx \\
= & \ y' a_2 v|_\alpha^\beta - \int_\alpha^\beta y' (a_2 v)'\, dx \\
& + y a_1 v|_\alpha^\beta - \int_\alpha^\beta y (a_1 v)'\, dx + \int_\alpha^\beta y a_0 v\, dx \\
= & \ [y' a_2 v + y a_1 v - y (a_2 v)']|_\alpha^\beta + \int_\alpha^\beta y \left[(a_2 v)'' - (a_1 v)' + a_0 v \right] dx \\
= & \ [a_2 (y' v - v' y) + (a_1 - a_2') y v]|_\alpha^\beta + \int_\alpha^\beta y \mathbb{M}(v)\, dx
\end{aligned}$$

where

$$\mathbb{M}(v) = a_2(x) v''(x) + (2a_2'(x) - a_1(x)) v'(x) + (a_2''(x) - a_1'(x) + a_0(x)) v(x) \ .$$

The operator $\mathbb{M}$ is called the *adjoint* of the operator $\mathbb{L}$.

Of particular importance in the theory of eigenvalues and eigenfunctions are operators which are *self-adjoint*; i.e. for which $\mathbb{L} = \mathbb{M}$. Comparing $\mathbb{L}$ and $\mathbb{M}$, we see that this happens if

$$a_2''(x) - a_1'(x) = 0 \ ; \ 2a_2'(x) - a_1(x) = a_1(x) \ ; \quad \text{i.e. if} \ \ a_1(x) = a_2'(x) \ .$$

In this case

$$\mathbb{L}(y) = (a_2(x) y'(x))' + a_0(x) y(x)$$

and

$$\int_\alpha^\beta \mathbb{L}(y) v\, dx = \int_\alpha^\beta y \mathbb{L}(v)\, dx \ + \ a_2(x)(v(x) y'(x) - y(x) v'(x))|_\alpha^\beta \ .$$

Self-adjoint operators and orthogonality.

Let $\mathbb{L}$ be a self-adjoint operator, and let y_m, y_n be functions such that

$$\mathbb{L}(y_m) = \lambda_m q(x) y_m(x) \ ; \quad \mathbb{L}(y_n) = \lambda_n q(x) y_n(x) \ ; \ \lambda_m \neq \lambda_n \ .$$

Then

$$\begin{aligned}\lambda_m \int_\alpha^\beta q(x) y_m(x) y_n(x)\, dx &= \int_\alpha^\beta \mathbb{L}(y_m) y_n \, dx \\ &= \int_\alpha^\beta y_m \mathbb{L}(y_n)\, dx \ + \ a_2(x)(y_n(x)y_m'(x) - y_m(x)y_n'(x))\big|_\alpha^\beta \\ &= \lambda_n \int_\alpha^\beta q(x) y_m(x) y_n(x)\, dx \\ &\qquad + \ a_2(x)(y_n(x)y_m'(x) - y_m(x)y_n'(x))\big|_\alpha^\beta\end{aligned}$$

$$(\lambda_m - \lambda_n) \int_\alpha^\beta q(x) y_m(x) y_n(x)\, dx = \ a_2(\beta) \begin{vmatrix} y_n(\beta) & y_n'(\beta) \\ y_m(\beta) & y_m'(\beta) \end{vmatrix} - a_2(\alpha) \begin{vmatrix} y_n(\alpha) & y_n'(\alpha) \\ y_m(\alpha) & y_m'(\alpha) \end{vmatrix}$$

The terms on the right hand side vanish if

(a)

$$\begin{gathered} a_2(\beta) = 0 \\ \text{or} \quad \exists\ (k_{21}, k_{22}) \ \neq \ (0,0) \ \text{ such that} \\ k_{21} y_n(\beta) + k_{22} y_n'(\beta) = 0 \\ k_{21} y_m(\beta) + k_{22} y_m'(\beta) = 0 \\ \text{and} \end{gathered}$$

(b)

$$\begin{gathered} a_2(\alpha) = 0 \\ \text{or} \quad \exists\ (k_{11}, k_{12}) \ \neq \ (0,0) \ \text{ such that} \\ k_{11} y_n(\alpha) + k_{12} y_n'(\alpha) = 0 \\ k_{11} y_m(\alpha) + k_{12} y_m'(\alpha) = 0 \ . \end{gathered}$$

When this occurs, we have

$$\int_\alpha^\beta q(x) y_m(x) y_n(x)\, dx = 0 \ ,$$

and we say that the functions y_m and y_n are orthogonal on $[\alpha, \beta]$ with respect to the *weight factor* $q(x)$.

We have already met examples of both types of orthogonality condition:

For Legendre polynomials, $a_2(x) = 1 - x^2$ vanishes at $x = \pm 1$, while for the earlier examples of eigenfunctions we have had $a_2(x) = 1$, which has no zeros, so that it has been the boundary conditions on the eigenfunctions which has ensured orthogonality.

Modifying the operator.

Since orthogonality is associated with self-adjoint operators, what can be done with operators which are not self-adjoint?

In the case of second order operators the answer is that we can render them self-adjoint by multiplying by a suitable function.

Given

$$\mathbb{L}(y) = a_2(x)y''(x) + a_1(x)y'(x) + a_0(x)y(x) \ ,$$

then

$$\mathbb{L}^*(y) \equiv p(x)\mathbb{L}(y) = p(x)a_2(x)y'' + p(x)a_1(x)y' + p(x)a_0(x)y$$

will be self-adjoint provided

$$\begin{aligned}
p(x)a_1(x) &= (p(x)a_2(x))' \\
&= p(x)a_2'(x) + p'(x)a_2(x) \\
p(x)(a_1(x) - a_2'(x)) &= p'(x)a_2(x) \\
\frac{p'(x)}{p(x)} &= \frac{a_1(x) - a_2'(x)}{a_2(x)} \\
\log(p(x)) &= \int \frac{a_1(x)}{a_2(x)}\,dx - \log(a_2(x)) \\
p(x)a_2(x) &= \exp\left(\int \frac{a_1(x)}{a_2(x)}\,dx\right) \ .
\end{aligned}$$

Example continued.

The equation for the eigenfunctions was

$$X'' + X' + \lambda X = 0 \ .$$

Here $a_2 = 1$, $a_1 = 1$ so that $p(x) = e^x$. Hence we have

$$\begin{aligned}
e^x X'' + e^x X' &= -\lambda e^x X \\
(e^x X')' &= -\lambda e^x X \\
(e^x X_m')' &= -\lambda_m e^x X_m \\
(e^x X_n')' &= -\lambda_n e^x X_n \\
\lambda_m \int_0^1 e^x X_m X_n\,dx &= -\int_0^1 (e^x X_m')' X_n\,dx \\
&= \int_0^1 e^x X_m' X_n'\,dx - e^x X_m'(x)X_n(x)\Big|_0^1 \\
&= \lambda_n \int_0^1 e^x X_m X_n\,dx \ ,
\end{aligned}$$

so that X_m and X_n are orthogonal on $[0,1]$ with weight factor e^x.

$$\begin{aligned}
\text{If}\qquad 1 &= \sum_{m=0}^{\infty} a_m X_m(x) \\
\int_0^1 1.e^x X_n\,dx &= \sum_{m=0}^{\infty} a_m \int_0^1 e^x X_m X_n\,dx \\
&= a_n \int_0^1 e^x X_n^2\,dx \\
a_n &= \int_0^1 e^x X_n(x)\,dx \Bigg/ \int_0^1 e^x X_n^2(x)\,dx \ .
\end{aligned}$$

$$\begin{aligned}
\int_0^1 e^x X_n^2(x)\,dx &= \int_0^1 \sin^2(\omega_n x)\,dx \\
&= \frac{1}{2}\int_0^1 (1-\cos(2\omega_n x))\,dx \\
&= \frac{1}{2}\left[x - \frac{1}{2\omega_n}\sin(2\omega_n x)\right]_0^1 \\
&= \frac{1}{2} - \frac{1}{4\omega_n}\sin(2\omega_n) \\
&= \frac{1}{2} - \frac{1}{4\omega_n^2+1} = \frac{4\omega_n^2-1}{2(4\omega_n^2+1)}
\end{aligned}$$

$$\begin{aligned}
\int_0^1 e^{x/2}\sin(\omega_n x)\,dx &= \frac{1}{4\omega_n^2+1}\left[2e^{x/2}\sin(\omega_n x) - 4\omega_n e^{x/2}\cos(\omega_n x)\right]_0^1 \\
&= \frac{1}{4\omega_n^2+1}\left[2e^{1/2}(\sin(\omega_n) - 2\omega_n\cos(\omega_n)) + 4\omega_n\right] \\
&= \frac{4\omega_n}{4\omega_n^2+1}
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{8\omega_n}{4\omega_n^2-1} \\
u(x,t) &= \sum_{n=0}^{\infty} \frac{8\omega_n}{4\omega_n^2-1} e^{-\lambda_n t} e^{-x/2}\sin(\omega_n x) \ .
\end{aligned}$$

Fourier-Bessel expansions.

Consider the following equation, which relates to vibrations of a drum.

$$\begin{gathered}
\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \\
u(1,t) = 0 \\
u(r,0) = f(r)\ ;\ u_t(r,0) = g(r)
\end{gathered}$$

While the boundary condition is homogeneous, if we look for solutions in the form $u(r,t) = R(r)T(t)$ for the problem

$$\begin{gathered}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} \\ u(1,t) = 0\end{gathered}$$

there is only one condition on the function $R(r)$, namely $R(1) = 0$. However, the separated equation has a singular point at $r = 0$, and we obtain a second condition by requiring that the solution be finite when $r = 0$.

Substituting the form for u into the equation we have

$$\begin{gathered}R''(r)T + \frac{1}{r}R'(r)T = \frac{1}{c^2}R\ddot{T} \\ \frac{r^2R'' + rR'}{r^2R} = \frac{1}{c^2}\frac{\ddot{T}}{T} = -\lambda \\ r^2R'' + rR' + \lambda r^2 R = 0\end{gathered}$$

There are three cases to consider:

Case 1: $\lambda < 0$. Set $\lambda = -p^2$, $s = pr$, and $R(s/p) = S(s)$. The equation becomes

$$s^2S'' + sS' - s^2S = 0 \ ,$$

which has a regular singular point at $s = 0$. The indicial equation is

$$\rho(\rho - 1) + \rho - 0 = 0 \ ; \ \rho^2 = 0 \ ; \ \rho = 0, 0 \ .$$

Since we have repeated roots for the indicial equation, the second solution has logarithmic behaviour near $s = 0$, and hence can be ignored. Applying the method of Frobenius to the equation, it can be shown that any solution which is finite at $s = 0$ is a multiple of

$$S(s) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2}\left(\frac{s}{2}\right)^{2n} \ .$$

However, this function is obviously positive for all s, so that there are no values of p for which $R(1) = S(p) = 0$. Hence there are no eigenfunctions of this form.

Case 2: $\lambda = 0$. The equation reduces to the Euler equation

$$r^2R'' + rR' = 0 \ ,$$

for which the general solution is $R = a + b\log r$. The requirement that the solution be finite at the origin makes $b = 0$, and now $R(1) = 0$ ensures that $a = 0$ also, and that $\lambda = 0$ is not an eigenvalue.

Case 3: $\lambda > 0$ Set $\lambda = \omega^2$, $s = \omega r$, and $R(s/\omega) = S(s)$. The equation becomes

$$s^2S'' + sS' + s^2S = 0 \ ,$$

which is Bessel's equation of order zero. The general solution is

$$S(s) = aJ_0(s) + bY_0(s) \ ,$$

but we require $b = 0$ in order to have S finite at $s = 0$, so that our equation for the eigenvalues is

$$J_0(\omega) = 0\ .$$

The first six such values are

$$2.4048 \quad 5.5201 \quad 8.6537 \quad 11.7915 \quad 14.9309 \quad 18.0711\ .$$

[Mathematical Handbook: p250]

The eigenfunction $R_n(r) = J_0(\omega_n r)$ satisfies the equation

$$r^2 R_n'' + r R_n' = -\omega_n^2 r^2 R_n\ ,$$

which in self-adjoint form is

$$(r R_n')' = -\omega_n^2 r R_n\ ,$$

so that the orthogonality relation is

$$\int_0^1 r R_n(r) R_m(r)\, dr = 0\ ;\ m \neq n\ .$$

Note that in this case the weight function ensures that there is no contribution from the origin, while the boundary condition takes care of the contribution from $r = 1$.

When $\lambda = \omega_n^2$, the function $T(t)$ satisfies

$$\ddot{T}(t) + \omega_n^2 c^2 T(t) = 0 \quad ; \quad T(t) = a_n \cos(c\omega_n t) + b_n \sin(c\omega_n t)\ .$$

The general solution for the problem is therefore

$$u(r,t) = \sum_{n=1}^{\infty} (a_n \cos(c\omega_n t) + b_n \sin(c\omega_n t)) J_0(\omega_n r)\ .$$

The coefficients a_n and b_n are recovered from the initial conditions via

$$\begin{aligned}
f(r) &= \sum_{n=1}^{\infty} a_n J_0(\omega_n r) \\
a_n &= \int_0^1 r f(r) J_0(\omega_n r)\, dr \Big/ \int_0^1 r J_0^2(\omega_n r)\, dr \\
g(r) &= \sum_{n=1}^{\infty} c\omega_n b_n J_0(\omega_n r) \\
b_n &= \frac{1}{c\omega_n} \int_0^1 r g(r) J_0(\omega_n r)\, dr \Big/ \int_0^1 r J_0^2(\omega_n r)\, dr\ .
\end{aligned}$$

Expansions of this form are common in problems with cylindrical symmetry.

The expansion

$$f(r) = \sum_{n=1}^{\infty} a_n J_0(\omega_n r)$$

is called a *Fourier-Bessel* expansion.

EXAMPLES OF EXAM QUESTIONS

1. Consider the boundary value problem

$$y'' + \lambda y = 0 \; ; \quad y(0) + y'(0) = 0 \; , \quad y'(1) = 0 \; .$$

Show that, if y_m and y_n are eigenfunctions corresponding to the eigenvalues λ_m and λ_n respectively, with $\lambda_m \neq \lambda_n$, then

$$\int_0^1 y_m(x) y_n(x) \, dx = 0 \; .$$

2. Determine the eigenvalues and eigenfunctions for the boundary value problem

$$y'' + 2y' + \lambda y = 0 \; ; \quad y'(0) = 0 \; , \quad y(1) = 0 \; ,$$

and derive an orthogonality relation between eigenfunctions corresponding to distinct eigenvalues.

3. Find the solution $u(x, y)$ of the following boundary value problem in the rectangle $0 < x < 2$, $0 < y < 1$.

$$\begin{gathered} u_{xx} + u_{yy} = 0 \\ u(0, y) = u(2, y) = 0 \; , \quad 0 < y < 1 \\ u(x, 0) = 0 \; , \; u(x, 1) = 1 - |x - 1| \; , \quad 0 \leq x \leq 2 \end{gathered}$$

4. Find the solution $u(x, t)$ of the initial value problem

$$\begin{gathered} u_{xx} = u_{tt} \\ u(0, t) = 0 \; , \quad u_x(1, t) = 0 \\ u(x, 0) = 0 \; , \quad u_t(x, 0) = \frac{x}{2} - \frac{x^2}{4} \end{gathered}$$