Solutions of second order linear differential equations as power series

Suppose that we are trying to solve the initial value problem

$$f_2(x)y'' + f_1(x)y' + f_0(x)y = 0 \ ; \quad y(x_0) = a_0 \ ; \ y'(x_0) = a_1 \ ,$$

where the functions f_i are continuous in some interval containing x_0.

Substituting the initial values in the equation, we find

$$f_2(x_0)y''(x_0) + f_1(x_0)y'(x_0) + f_0(x_0)y(x_0) = 0 \ ,$$

and, provided $f_2(x_0) \neq 0$,

$$y''(x_0) = -(f_1(x_0)a_1 + f_0(x_0)a_0)/f_2(x_0) \ .$$

If the functions f_0, f_1, and f_2 are differentiable, we can differentiate the differential equation to obtain

$$f_2(x)y''' + (f_2'(x) + f_1(x))y'' + (f_1'(x) + f_0(x))y' + f_0'(x)y = 0 \ .$$

Once again, if we evaluate this equation at x_0, we obtain

$$\begin{aligned} f_2(x_0)y'''(x_0) + (f_2'(x_0) + f_1(x_0))y''(x_0) + (f_1'(x_0) + f_0(x_0))y'(x_0) + f_0'(x_0)y(x_0) = 0 \\ y'''(x_0) = (f_2'(x_0) + f_1(x_0))(f_1(x_0)a_1 + f_0(x_0)a_0)/f_2^2(x_0) \\ - \left((f_1'(x_0) + f_0(x_0))a_1 + f_0'(x_0)a_0\right)/f_2(x_0) \end{aligned}$$

We could proceed in this fashion for as many times as the coefficient functions have derivatives. Indeed, if the coefficient functions are analytic functions, and have derivatives of all orders, we could in theory determine the values of all the derivatives of the solution y at the point x_0. This means that we could (again, in theory) write down the Taylor series expansion

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n y}{dx^n}(x_0)(x - x_0)^n \ .$$

It can be proved that this Taylor series expansion converges for $|x - x_0| < R$, where R is the distance from x_0 to the nearest (complex) zero of $f_2(x)$.

Note that if we write the equations in the standard form

$$y'' + p_1(x)y' + p_0(x)y = 0 \ ,$$

we have $p_1 = f_1/f_2$ and $p_0 = f_0/f_2$, so that as long as $f_2(x) \neq 0$, p_1 and p_2 are continuous, and the existence and uniqueness theorems apply.

Independent solutions.

The procedure outlined is valid for any choice of the initial values a_0 and a_1. In particular, we can determine a solution $y_1(x)$ such that $y_1(x_0) = 1$, $y_1'(x_0) = 0$, and another solution $y_2(x)$ such that $y_2(x_0) = 0$, $y_2'(x_0) = 1$. For these solutions we have

$$W(y_1, y_2)(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$

so that they are independent solutions in a neighbourhood of x_0, and the general solution satisfying $y(x_0) = a_0, y'(x_0) = a_1$ can be written as

$$y(x) = a_0 y_1(x) + a_1 y_2(x) \ .$$

Practical solution techniques.

In practice we restrict our attention to the cases in which the functions f_i are polynomials in x, of degree at most 2.

Furthermore, rather than differentiate the equation, we assume a power series form for the solution y and substitute this into the equation.

e.g. Proceeding formally, find two linearly independent power series solutions in powers of x for each of the following differential equations.

$$y'' - xy' - y = 0$$

$$\text{Let } \; y = \sum_{n=0}^{\infty} a_n x^n \; ; \; y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \; ; \; xy' = \sum_{n=0}^{\infty} n a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n$$

$$y'' - xy' - y = \sum_{n=0}^{\infty} x^n((n+2)(n+1)a_{n+2} - na_n - a_n)$$

which vanishes provided $\quad (n+2)(n+1)a_{n+2} = (n+1)a_n \quad$ for all $\; n \geq 0$.

Since this recurrence relation does not involve a_{n+1}, we can determine the odd and even terms independently. Setting $n = 2m$, we have

$$(2m+2)a_{2m+2} = a_{2m} \; ; \; a_{2(m+1)} = \frac{1}{2}\frac{1}{m+1}a_{2m}$$

Considering the first few terms we have

$$a_2 = \frac{1}{2}\frac{1}{1}a_0 \; ; \; a_4 = \frac{1}{2}\frac{1}{2}a_2 = \left(\frac{1}{2}\right)^2 \frac{1}{1.2}a_0$$

$$a_6 = \frac{1}{2}\frac{1}{3}a_4 = \left(\frac{1}{2}\right)^3 \frac{1}{1.2.3}a_0 = \left(\frac{1}{2}\right)^3 \frac{1}{3!}a_0$$

$$\text{If } \; a_{2m} = \left(\frac{1}{2}\right)^m \frac{1}{m!}a_0,$$

$$a_{2(m+1)} = \frac{1}{2}\frac{1}{m+1}\left(\frac{1}{2}\right)^m \frac{1}{m!}a_0 = \left(\frac{1}{2}\right)^{m+1} \frac{1}{(m+1)!}a_0.$$

Since this formula holds for $m = 0$, it holds for all $m \geq 0$ by the principle of Mathematical Induction.

One solution of this equation is therefore

$$y = a_0 \sum_{m=0}^{\infty} \left(\frac{1}{2}\right)^m \frac{1}{m!}x^{2m} = a_0 \sum_{m=0}^{\infty} \frac{1}{m!}\left(\frac{x^2}{2}\right)^m = a_0 e^{x^2/2}.$$

[Note that this power series converges for all values of x. This is in accord with the earlier remark, since the function $f_2(x) \equiv 1$ has no zeros.]

For a second independent solution we set $n = 2m+1$, giving

$$(2m+3)a_{2m+3} = a_{2m+1} \; ; \; a_{2(m+1)+1} = \frac{1}{2m+3}a_{2m+1}$$

Again considering the first few terms, we have

$$a_3 = \frac{1}{3}a_1 \ ; \ a_5 = \frac{1}{3.5}a_1 \ ; \ a_7 = \frac{1}{3.5.7}a_1.$$

and in general

$$a_{2m+1} = \frac{1}{1.3.5.\dots(2m+1)}a_1 = \frac{2.4.\dots 2m}{(2m+1)!}a_1 = \frac{2^m m!}{(2m+1)!}a_1.$$

A second solution is therefore

$$y = a_1 \sum_{m=0}^{\infty} \frac{2^m m!}{(2m+1)!} x^{2m+1}.$$

Using the ratio test, we see that the radius of convergence of this series is

$$\lim_{m\to\infty} \frac{2^m m!}{(2m+1)!} \frac{(2m+3)!}{2^{m+1}(m+1)!} = \lim_{m\to\infty} \frac{(2m+2)(2m+3)}{2(m+1)} = \infty \ ,$$

which also accords with our expectations.

e.g.

$$(1-x)y'' + y = 0$$

$$\text{Let} \quad y = \sum_{n=0}^{\infty} a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}x^n$$

$$xy'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-1} = \sum_{n=1}^{\infty} n(n-1)a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)na_{n+1}x^n$$

$$(1-x)y'' + y = \sum_{n=0}^{\infty} x^n((n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + a_n)$$

which vanishes provided $(n+2)(n+1)a_{n+2} = (n+1)na_{n+1} - a_n$ for all $n \geq 0$.

We can determine two independent solutions by first considering $a_0 = 1, a_1 = 0$, and then $a_0 = 0, a_1 = 1$.

In the first case we have

$$a_2 = \frac{1}{2}(0.0-1) = -\frac{1}{2} \ ; \ a_3 = \frac{1}{6}(2(-\frac{1}{2}) - 0) = -\frac{1}{6}$$

$$a_4 = \frac{1}{12}(6(-\frac{1}{6}) + \frac{1}{2}) = -\frac{1}{24} \ ; \ a_5 = \frac{1}{20}(12(-\frac{1}{24}) + \frac{1}{6}) = -\frac{1}{60}$$

$$y_1 = 1 - \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{60}x^5 - \dots$$

In the second case we have

$$a_2 = \frac{1}{2}(0.1 - 0) = 0 \ ; \ a_3 = \frac{1}{6}(2.0 - 1) = -\frac{1}{6}$$
$$a_4 = \frac{1}{12}(6(-\frac{1}{6}) - 0) = -\frac{1}{12} \ ; \ a_5 = \frac{1}{20}(12(-\frac{1}{12}) + \frac{1}{6}) = -\frac{1}{24}$$
$$y_2 = x - \frac{1}{6}x^3 - \frac{1}{12}x^4 - \frac{1}{24}x^5 - \ldots$$

Note that in this case we do not have nice closed forms for the coefficients, so that we have to content ourselves with the numeric computation of the leading terms.

Since $f_2(x) = 1 - x$ has a zero at $x = 1$, these power series will converge for $|x| < 1$.

Legendre's Polynomials.

One important equation of this type, which arises in three-dimensional problems using spherical polar co-ordinates, is *Legendre's equation*

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0 \ .$$

If we derive solutions as power series in x, we expect that the series will converge for $|x| < 1$.

Proceeding as above, we have;

$$\text{Let} \ \ y = \sum_{r=0}^{\infty} a_r x^r \ ; \ y' = \sum_{r=0}^{\infty} r a_r x^{r-1} \ ; \ xy' = \sum_{r=0}^{\infty} r a_r x^r$$
$$y'' = \sum_{r=0}^{\infty} r(r-1) a_r x^{r-2} = \sum_{r=0}^{\infty} (r+2)(r+1) a_{r+2} x^r \ ; \ x^2 y'' = \sum_{r=0}^{\infty} r(r-1) a_r x^r$$
$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y$$
$$= \sum_{r=0}^{\infty} x^r ((r+2)(r+1)a_{r+2} - r(r-1)a_r - 2ra_r + \alpha(\alpha+1)a_r)$$

which vanishes if the coefficients satisfy the recurrence relation

$$(r+2)(r+1)a_{r+2} = (r - \alpha)(r + \alpha + 1)a_r.$$

Since the recurrence relation does not involve a_{r+1}, we can find two independent solutions one of which is even (i.e. $a_r = 0$ if r is odd) and the other of which is odd (i.e. $a_r = 0$ if r is even).

For the even solution, we set $a_0 = 1$, $a_1 = 0$.

Note that, since $a_1 = 0$, $a_3 = 0$, and hence $a_5 = 0$ etc.

For the other coefficients, set $r = 2m$ in the recurrence relation.

$$(2m+2)(2m+1)a_{2(m+1)} = (2m - \alpha)(2m + 1 + \alpha)a_{2m} \ .$$

The calculations can be further simplified if we set $a_{2m} = b_{2m}/(2m)!$. Then

$$\begin{aligned} b_{2(m+1)} &= (2m-\alpha)(2m+1+\alpha)b_{2m} \\ b_0 &= 1 \\ b_2 &= (-\alpha)(1+\alpha) \\ b_4 &= (-\alpha)(2-\alpha)(1+\alpha)(3+\alpha) \\ b_{2m} &= \prod_{k=0}^{m-1}(2k-\alpha)\prod_{k=0}^{m-1}(2k+1+\alpha) \end{aligned}$$

Notice that if $\alpha = 2N$, where N is an non-negative integer, then the factor $(2m-\alpha)$ vanishes when $m = N$, so that b_{2m} (and a_{2m}) also vanish for $m > N$. In this case the power series reduces to a polynomial. For example

$$\begin{aligned} \alpha = 0\ ; &\quad y = 1 \\ \alpha = 2\ ; &\quad y = 1 - 3x^2 \\ \alpha = 4\ ; &\quad y = 1 - 10x^2 + \frac{35}{3}x^4 \end{aligned}$$

Scaling these solutions so that they take the value 1 when $x = 1$, we obtain the *Legendre Polynomials*, designated $P_{2N}(x)$. That is

$$\begin{aligned} P_0(x) &= 1 \\ P_2(x) &= \frac{3}{2}x^2 - \frac{1}{2} \\ P_4(x) &= \frac{35}{8}x^4 - \frac{15}{4}x^2 + \frac{3}{8} \end{aligned}$$

For the odd solution, set $a_0 = 0$, $a_1 = 1$, and set

$$r = 2m+1\ , \qquad \text{and} \qquad a_{2m+1} = b_{2m+1}/(2m+1)!$$

in the recurrence relation. This gives

$$\begin{aligned} b_{2m+3} &= (2m+1-\alpha)(2m+2+\alpha)b_{2m+1} \\ b_1 &= 1 \\ b_3 &= (1-\alpha)(2+\alpha) \\ b_5 &= (1-\alpha)(3-\alpha)(2+\alpha)(4+\alpha) \\ b_{2m+1} &= \prod_{k=0}^{m-1}(2k+1-\alpha)\prod_{k=1}^{m}(2k+\alpha) \end{aligned}$$

In this case the solution reduces to a polynomial if $\alpha = 2N+1$, where N is a non-negative integer. For example

$$\begin{aligned} \alpha = 1\ ; &\quad y = x \\ \alpha = 3\ ; &\quad y = x - \frac{5}{3}x^3 \\ \alpha = 5\ ; &\quad y = x - \frac{14}{3}x^3 + \frac{21}{5}x^5 \end{aligned}$$

The corresponding Legendre polynomials $P_{2N+1}(x)$ are the solutions for which $P_{2N+1}(1) = 1$, namely

$$\begin{aligned} P_1(x) &= x \\ P_3(x) &= \frac{5}{2}x^3 - \frac{3}{2}x \\ P_5(x) &= \frac{63}{8}x^5 - \frac{35}{4}x^3 + \frac{15}{8} \end{aligned}$$

Legendre Polynomials and Orthogonality.

Legendre's differential equation can be written in the form

$$\left((1-x^2)y'\right)' = -\alpha(\alpha+1)y \ ,$$

so that the Legendre Polynomial $P_n(x)$ satisfies

$$n(n+1)P_n(x) = -\frac{d}{dx}\left((1-x^2)P_n'(x)\right) \ .$$

Therefore

$$\begin{aligned} n(n+1)\int_{-1}^{1} P_n(x)P_m(x)\,dx &= -\int_{-1}^{1}\left[\frac{d}{dx}\left((1-x^2)P_n'(x)\right)\right]P_m(x)\,dx \\ &= -\left((1-x^2)P_n'(x)\right)P_m(x)\Big|_{-1}^{1} + \int_{-1}^{1}(1-x^2)P_n'(x)P_m'(x)\,dx \\ &= \int_{-1}^{1}(1-x^2)P_m'(x)P_n'(x)\,dx \\ &= m(m+1)\int_{-1}^{1}P_m(x)P_n(x)\,dx \end{aligned}$$

$\blacksquare$

Hence, if $m \neq n$, we have

$$\int_{-1}^{1} P_n(x)P_m(x)\,dx = 0 \ .$$

The Legendre Polynomials form a set of *Orthogonal Polynomials* on the interval $[-1, 1]$.

This property is important in the theory of approximation. If we want the best polynomial approximation $p(x)$ of degree N to a function $f(x)$ on the interval $[-1, 1]$ in the sense of minimizing the *Mean Square deviation*

$$\int_{-1}^{1} \left(f(x) - p(x)\right)^2\, dx$$

then it can be shown that

$$p(x) = \sum_{k=0}^{N} a_k P_k(x)$$

where the coefficients a_k are calculated by

$$\int_{-1}^{1} f(x)P_k(x)\,dx = a_k \int_{-1}^{1} P_k^2(x)\,dx \quad \left(= \frac{2}{2k+1}a_k\right) \ .$$

[Similar considerations give rise to the Fourier Series expansions.]

The orthogonality of these polynomials is also relevant in the numerical procedure known as *Gaussian Quadrature*.

Singular points.

Suppose that the functions f_0, f_1, and f_2 are analytic functions on the real line $(-\infty, \infty)$. (As noted above, they are usually polynomials.) Then we can find power series solutions of the differential equation

$$f_2(x)y'' + f_1(x)y' + f_0(x)y = 0$$

in powers of $(x - x_0)$ provided $f_2(x_0) \neq 0$. Such points are called ordinary points of the differential equation, while the points at which $f_2(x) = 0$ are called singular points of the equation.

Consider the differential equation

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \ .$$

This equation has one (finite) singular point at $x = 0$. Therefore, for any $c \neq 0$ we can find power series solutions of the form

$$\sum_{k=0}^{\infty} a_k(x - c)^k \ .$$

However, these series converge only for $|x - c| < |c|$, so that no solution is valid for all $0 < |x| < \infty$. This problem could be overcome if we could somehow incorporate the singular point into the power series.

It turns out that this is possible if the singular point is *regular*. A regular singular point is one for which the equation can be written in the form

$$F_2(x)(x - c)^2y'' + F_1(x)(x - c)y' + F_0(x)y = 0$$

where $F_0(c)$ and $F_1(c)$ are finite and $F_2(c) \neq 0$.

In this case we can consider the approximate equation

$$F_2(c)(x - c)^2y'' + F_1(c)(x - c)y' + F_0(c)y = 0$$

or, equivalently

$$(x - c)^2y'' + \alpha(x - c)y' + \beta y = 0$$

in the neighbourhood of $x = c$, and use this information to construct a generalised power series solution on a wider interval.

The approximate equation is in the form of an *Euler equation*, the solutions for which were considered earlier in the case of systems.

Euler equations.

For ease of writing, it will be assumed that $c = 0$.

Consider the differential equation

$$\mathbb{L}(y) \ \equiv \ x^2y'' + \alpha xy' + \beta y = 0 \ ; \quad x > 0 \ .$$

$$\begin{aligned} \mathbb{L}(x^r) &= x^2(r(r-1)x^{r-2}) + \alpha x(rx^{r-1}) + \beta x^r \\ &= (r^2 - r + \alpha r + \beta)x^r \\ &= F(r)x^r \end{aligned}$$

$$\text{where} \quad F(r) = r^2 + (\alpha - 1)r + \beta \ .$$

$F(r)$ is called the *indicial polynomial* for the operator $\mathbb{L}$, and we can see that x^a will be a solution of the equation if and only if $F(a) = 0$. As before, the solutions for $x < 0$ can be obtained by replacing x by $|x|$ if necessary.

e.g. Determine the general solution of each of the following second order differential equations that is valid in any interval not including the singular point.

(1)
$$x^2y'' + 4xy' + 2y = 0$$

Let $y = x^r$. $y' = rx^{r-1}, y'' = r(r-1)x^{r-2}$.

$$x^2y'' + 4xy' + 2y = r(r-1)x^r + 4rx^r + 2x^r = (r^2 - r + 4r + 2)x^r = (r^2 + 3r + 2)x^r$$

The indicial equation is $(r^2 + 3r + 2) = 0$; $(r + 2)(r + 1) = 0$; $r = -1$ or -2.

Therefore the general solution of the equation is

$$y = c_1x^{-1} + c_2x^{-2},$$

which is valid in either $(-\infty, 0)$ or $(0, \infty)$.

(10)
$$(x-2)^2y'' + 5(x-2)y' + 8y = 0$$

Let $y = (x-2)^r$. $y' = r(x-2)^{r-1}, y'' = r(r-1)(x-2)^{r-2}$.

$$(x-2)^2y'' + 5(x-2)y' + 8y = (r(r-1) + 5r + 8)(x-2)^r = (r^2 + 4r + 8)(x-2)^r.$$

The indicial equation is $r^2 + 4r + 8 = 0$; $(r + 2)^2 = -4$; $r = -2 \pm 2i$.

The equation therefore has solutions of the form

$$\begin{aligned} y &= |x-2|^{-2}\,|x-2|^{2i} = |x-2|^{-2}\exp\left(2i\log|x-2|\right) \\ &= |x-2|^{-2}\left(\cos\left(2\log|x-2|\right) + i\sin\left(2\log|x-2|\right)\right) \end{aligned}$$

together with the complex conjugate. The general solution can therefore be written as

$$y = |x-2|^{-2}\left(c_1\cos\left(2\log|x-2|\right) + c_2\sin\left(2\log|x-2|\right)\right),$$

and the solution is valid on either $(-\infty, 2)$ or $(2, \infty)$.

When the indicial polynomial has repeated roots; $F(r) = (r-a)^2$; there is only one independent solution in this simple form. To obtain a second solution we differentiate with respect to the parameter r.

$$\begin{aligned} \mathbb{L}\left(\frac{\partial}{\partial r}x^r\right) &= \frac{\partial}{\partial r}F(r)x^r \\ &= 2(r-a)x^r + (r-a)^2\frac{\partial}{\partial r}x^r \\ &= 0 \quad \text{if} \quad r = a\ . \\ \frac{\partial}{\partial r}x^r &= \frac{\partial}{\partial r}e^{r\log x} \\ &= \log x\, e^{r\log x} \ = \ \log x\, x^r\ . \end{aligned}$$

Therefore a second solution in the case where the indicial polynomial has a repeated root is $x^a\log x$ when $x > 0$, and $|x|^a\log|x|$ when $x < 0$.

e.g.

Find the general solution of

$$x^2y'' - xy' + y = 0 \ .$$

The indicial polynomial is

$$F(r) = r(r-1) - r + 1 = r^2 - 2r + 1 = (r-1)^2 \ .$$

Therefore the general solution of the equation is

$$y = \begin{cases} x(a_1 + a_2 \log x) & x > 0 \\ x(b_1 + b_2 \log |x|) & x < 0 \end{cases}$$

Alternatively we can use the substitution $y = vx^{r_1}$ to show that if r_1 is a repeated root of $r(r-1) + \alpha r + \beta = 0$, then x^{r_1} and $x^{r_1} \ln x$ are solutions of

$$x^2y'' + \alpha xy' + \beta y = 0, \qquad x > 0.$$

If the indicial equation has a repeated root, then

$$r(r-1) + \alpha r + \beta = (r - r_1)^2 \ ; \ \beta = r_1^2 \ ; \ \alpha - 1 = -2r_1 \ ; \ \alpha = 1 - 2r_1.$$

Setting $y = vx^{r_1}$, we have

$$y' = v'x^{r_1} + r_1 v x^{r_1 - 1} \ ; \ y'' = v''x^{r_1} + 2r_1 v' x^{r_1-1} + r_1(r_1-1)vx^{r_1-2}.$$

$$\begin{aligned} &x^2y'' + \alpha xy' + \beta y \\ &= x^2(v''x^{r_1} + 2r_1v'x^{r_1-1} + r_1(r_1-1)vx^{r_1-2}) + (1-2r_1)x(v'x^{r_1} + r_1vx^{r_1-1}) + r_1^2vx^{r_1} \\ &= v''x^{r_1+2} + v'x^{r_1+1}(2r_1 + 1 - 2r_1) + vx^{r_1}(r_1(r_1-1) + (1-2r_1)r_1 + r_1^2) \\ &= x^{r_1+1}(xv'' + v') \ = \ 0 \quad \text{if} \quad xv'' + v' = 0 \ ; \ (xv')' = 0 \\ &xv' = c_1 \ ; \ v' = c_1/x \ ; \ v = c_1 \log x + c_2 \ ; \ y = x^{r_1}(c_1 \log x + c_2). \end{aligned}$$

▮

The method of Frobenius.

For the case of a more general regular singular point, we now assume that there will be solutions of the form

$$y = (x-c)^r \sum_{k=0}^{\infty} a_k (x-c)^k \ = \ \sum_{k=0}^{\infty} a_k (x-c)^{k+r}$$

where it is further assumed that $a_0 \neq 0$.

Substituting this form into the equation

$$(x-c)^2y'' + (x-c)P(x)y' + Q(x)y = 0$$

we obtain

$$\begin{aligned}
y &= \sum_{k=0}^{\infty} a_k (x-c)^{k+r} \\
y' &= \sum_{k=0}^{\infty} (k+r) a_k (x-c)^{k+r-1} \\
(x-c)y' &= \sum_{k=0}^{\infty} (k+r) a_k (x-c)^{k+r} \\
y'' &= \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k (x-c)^{k+r-2} \\
(x-c)^2 y'' &= \sum_{k=0}^{\infty} (k+r)(k+r-1) a_k (x-c)^{k+r} \\
\text{Expanding} \quad P(x) &= \sum_{k=0}^{\infty} P_k (x-c)^k \\
\text{and} \qquad Q(x) &= \sum_{k=0}^{\infty} Q_k (x-c)^k \\
\text{we have} \quad \mathbb{L}(y) &= (x-c)^2 y'' + (x-c)P(x)y' + Q(x)y \\
&= (x-c)^r \left(\sum_{k=0}^{\infty} (k+r)(k+r-1)(x-c)^k \right. \\
&\quad + \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} P_{k-l}(l+r) a_l \right) (x-c)^k \\
&\quad \left. + \sum_{k=0}^{\infty} \left(\sum_{l=0}^{k} Q_{k-l} a_l \right) (x-c)^k \right) \\
&= (x-c)^r (r(r-1) + P_0 r + Q_0) a_0) + \\
&\quad (x-c)^r \sum_{k=1}^{\infty} (((k+r)(k+r-1) + P_0(k+r) + Q_0) a_k + \\
&\quad \sum_{l=0}^{k-1} ((l+r)P_l + Q_l) a_l)(x-c)^k \\
&= F(r) a_0 (x-c)^r + \\
&\quad \sum_{k=1}^{\infty} \left(F(k+r) a_k + \sum_{l=0}^{k-1} (P_{k-l}(l+r) + Q_{k-l}) a_l \right) (x-c)^{k+r}
\end{aligned}$$

This series expansion will be a solution of the equation provided

$$\begin{aligned}
F(r) a_0 &= 0 \\
F(k+r) a_k &= -\sum_{l=0}^{k-1} ((l+r) P_{k-l} + Q_{k-l}) a_l
\end{aligned}$$

The first condition gives us a quadratic equation $F(r) = 0$ for the appropriate indices $r = r_1, r_2$, and the second condition permits us to calculate the coefficients a_k as multiples of a_0 provided there is no value $N > 0$ such that $F(N + r_i) = 0$.

If $r_1 > r_2$, this can only happen if $r_1 - r_2 = N$, and then it only affects the calculation of the solution corresponding to $r = r_2$.

e.g. Each of the following differential equations has a regular singular point at $x = 0$. Determine the indicial equation, the recurrence relation, and the roots of the indicial equation. Find the series solution ($x > 0$) corresponding to the larger root. If the roots are unequal and do not differ by an integer, find the series solution corresponding to the smaller root.

(1) $$2xy'' + y' + xy = 0$$

$$\text{Setting} \quad y = \sum_{n=0}^{\infty} a_n x^{n+r} \ ; \ y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \ ; \ xy'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-1}$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+r+1} = \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}.$$

$$2xy'' + y' + xy = \sum_{n=0}^{\infty} [2(n+r)(n+r-1) + (n+r)] a_n x^{n+r-1} + \sum_{n=2}^{\infty} a_{n-2} x^{n+r-1}.$$

$$[2r(r-1) + r] a_0 = 0$$

$$[2(r+1)r + (r+1)] a_1 = 0$$

$$[2(n+r)(n+r-1) + (n+r)] a_n + a_{n-2} = 0 \quad \text{for} \quad n \geq 2$$

Since we assume that $a_0 \neq 0$, the first equation reduces to the indicial equation $r(2r-1) = 0$ whose roots are $r = 0$ and $r = 1/2$. The second equation is now $(r+1)(2r+1)a_1 = 0$. Since the polynomial in r is non-zero for both $r = 0$ and $r = 1/2$, we satisfy this equation by taking $a_1 = 0$. The recurrence relation now shows that $a_n = 0$ for all odd n.

When $r = 1/2$, and $n = 2m$, we have

$$2m(4m+1) a_{2m} = -a_{2(m-1)}$$

$$a_{2m} = -\frac{1}{2m(4m+1)} a_{2(m-1)} = \frac{(-1)^m}{2^m m!} \frac{a_0}{(4m+1)(4m-3)\ldots 5}$$

$$y = a_0 x^{1/2} \left(1 + \sum_{m=1}^{\infty} \frac{1}{m!(4m+1)\ldots 5} \left(\frac{-x^2}{2} \right)^m \right)$$

When $r = 0$ and $n = 2m$, we have

$$2m(4m-1)a_{2m} = -a_{2(m-1)}$$

$$a_{2m} = -\frac{1}{2m(4m-1)}a_{2(m-1)} = \frac{(-1)^m}{2^m m!}\frac{a_0}{(4m-1)(4m-5)\dots 3}$$

$$y = a_0\left(1+\sum_{m=1}^{\infty}\frac{1}{m!(4m-1)\dots 3}\left(\frac{-x^2}{2}\right)^m\right)$$

(2) $$xy'' + y = 0$$

$$\text{Setting} \quad y = \sum_{n=0}^{\infty} a_n x^{n+r} \ ; \ y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

$$xy'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-1} = \sum_{n=-1}^{\infty}(n+r+1)(n+r)a_{n+1}x^{n+r}$$

$$xy'' + y = r(r-1)a_0x^{r-1} + \sum_{n=0}^{\infty}[(n+r+1)(n+r)a_{n+1} + a_n]x^{n+r}$$

The indicial equation is $r(r-1) = 0$, whose roots $0, 1$ differ by an integer. Corresponding to the root $r = 1$, we have the recurrence relation

$$(n+2)(n+1)a_{n+1} = -a_n$$

$$a_{n+1} = -\frac{1}{(n+1)(n+2)}a_n = \frac{(-1)^{n+1}}{(n+1)!(n+2)!}a_0$$

$$y = a_0 x\sum_{n=0}^{\infty}\frac{(-x)^n}{n!(n+1)!}$$

e.g. The Laguerre (1834-1886) differential equation is

$$xy'' + (1-x)y' + \lambda y = 0.$$

Show that $x = 0$ is a regular singular point. Determine the indicial equation , its roots, the recurrence relation, and one solution ($x > 0$). Show that if $\lambda = m$, a positive integer, this solution reduces to a polynomial. When properly normalized this polynomial is known as the Laguerre polynomial, $L_m(x)$.

If $xy'' + (1-x)y' + \lambda y = 0$, $x^2y'' + (1-x)xy' + \lambda xy = 0$, $P(x) = (1-x) \to 1$ as $x \to 0$, and $Q(x) = \lambda x \to 0$ as $x \to 0$. Hence $x = 0$ is a regular singular point of the equation.

$$\text{Setting} \quad y = \sum_{n=0}^{\infty} a_n x^{n+r}$$

$$y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1} = \sum_{n=-1}^{\infty} (n+r+1) a_{n+1} x^{n+r}$$

$$y'' = \sum_{n=-1}^{\infty} (n+r+1)(n+r) a_{n+1} x^{n+r-1}$$

$$xy'' + (1-x)y' + \lambda y$$

$$= \sum_{n=-1}^{\infty} [(n+r+1)(n+r) + (n+r+1)] a_{n+1} x^{n+r} + \sum_{n=0}^{\infty} [-(n+r) + \lambda] a_n x^{n+r}$$

$$r^2 a_0 = 0$$

$$(n+r+1)^2 a_{n+1} = (n+r-\lambda) a_n \quad \text{for} \quad n \geq 0.$$

Since the indicial equation $r^2 = 0$ has the repeated root $r = 0$, we can determine only one solution from this approach. Setting $r = 0$ in the recurrence relation we have

$$(n+1)^2 a_{n+1} = (n-\lambda) a_n$$

$$a_{n+1} = \frac{(n-\lambda)}{(n+1)^2} a_n = \frac{(n-\lambda)\dots(-\lambda)}{((n+1)!)^2} a_0$$

$$y = a_0 \left(1 + \sum_{n=1}^{\infty} \frac{(n-1-\lambda)\dots(-\lambda)}{(n!)^2} x^n \right)$$

When $\lambda = m$ an integer, the product $(n-1-\lambda)\dots(-\lambda)$ contains the zero factor $(m-\lambda)$ whenever $n-1 \geq m$, $n \geq m+1$, so that the sum effectively terminates when $n = m$. Hence we have

$$y = a_0 \quad \text{when} \quad m = 0$$

$$y = a_0 \left(1 + \sum_{n=1}^{m} \frac{(-1)^n m(m-1)\dots(m-n+1)}{(n!)^2} x^n \right) \quad \text{when} \quad m \geq 1$$

The Laguerre polynomials are recovered by setting $a_0 = m!$, so that

$$L_0(x) = 1$$
$$L_1(x) = 1 - x$$
$$L_2(x) = 2 - 4x + x^2$$
$$L_3(x) = 6 - 18x + 9x^2 - x^3$$

Orthogonality.

If we multiply the differential equation by e^{-x}, we get

$$
\begin{aligned}
(xe^{-x}y')' &= -\lambda e^{-x}y \\
\text{so that} \quad \left(xe^{-x}L_m'(x)\right)' &= -me^{-x}L_m(x) \\
m\int_0^\infty e^{-x}L_m(x)L_n(x)\,dx &= -\int_0^\infty \left(xe^{-x}L_m'(x)\right)' L_n(x)\,dx \\
&= -\left. xe^{-x}L_m'(x)L_n(x)\right|_0^\infty + \int_0^\infty xe^{-x}L_m'(x)L_n'(x)\,dx \\
&= \int_0^\infty xe^{-x}L_n'(x)L_m'(x)\,dx \\
&= n\int_0^\infty e^{-x}L_n(x)L_m(x)\,dx \\
\int_0^\infty e^{-x}L_m(x)L_n(x)\,dx &= 0 \quad \text{if } m \neq n
\end{aligned}
$$

Hence, the Laguerre polynomials are orthogonal on the interval $[0,\infty)$ with respect to the weight factor e^{-x}. They are used to approximate functions on $[0,\infty)$, and in numerical integration. (c.f. Legendre polynomials)

Exceptional cases.

The standard Frobenius procedure allows the determination of two independent solutions in most cases. However, it gives only one solution when the indicial equation has repeated roots, and usually fails when the roots of the indicial equation differ by an integer.

Repeated roots.

When there are repeated roots of the indicial equation, we can obtain two independent solutions by effectively reversing the solution procedure. Having obtained

$$
\mathbb{L}(y) = F(r)a_0(x-c)^r + \sum_{k=1}^{\infty}\left(F(k+r)a_k + \sum_{l=0}^{k-1}(P_{k-l}(l+r)+Q_{k-l})a_l\right)(x-c)^{k+r}
$$

we formally set

$$
F(k+r)a_k = -\sum_{l=0}^{k-1}(P_{k-l}(l+r)+Q_{k-l})a_l
$$

and use this recurrence relation to calculate a_k as a function $a_k(r)$ of r. Then if we set

$$
y(x,r) = a_0(x-c)^r + \sum_{k=1}^{\infty} a_k(r)(x-c)^{k+r}
$$

we have

$$
\mathbb{L}(y(x,r)) = F(r)a_0(x-c)^r = (r-r_1)^2 a_0(x-c)^r \ .
$$

Hence

$$
\mathbb{L}(y(x,r_1)) = 0
$$

and
$$
\mathbb{L}\left(\left.\frac{\partial}{\partial r}y(x,r)\right|_{r_1}\right) = \left.\left[2(r-r_1)a_0(x-c)^r + (r-r_1)^2 a_0(x-c)^r\log(x-c)\right]\right|_{r_1} = 0
$$

so that we obtain two independent solutions of the equation.

e.g.

$$x^2y'' + xy' + 2xy = 0$$

The indicial equation for this differential equation is $r(r-1)+r+0=0$, $r^2=0$, so that in this case we have the repeated root $r=0$.

If we make the standard substitution

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \ , \ y' = \sum_{n=0}^{\infty}(n+r)a_n x^{n+r-1} \ , \ y'' = \sum_{n=0}^{\infty}(n+r)(n+r-1)a_n x^{n+r-2}$$

then

$$x^2y'' + xy' + 2xy = \sum_{n=0}^{\infty}[(n+r)(n+r-1)+(n+r)]a_n x^{n+r} + \sum_{n=1}^{\infty} 2a_{n-1}x^{n+r}$$

$$= r^2 a_0 x^r + \sum_{n=1}^{\infty}[(n+r)^2 a_n + 2a_{n-1}]x^{n+r}$$

$$= r^2 x^r \quad \text{provided} \quad a_0 = 1, \quad \text{and} \quad a_n = -\frac{2}{(n+r)^2}a_{n-1} \quad \text{for} \quad n \geq 1.$$

$$\text{This gives} \qquad a_n = \frac{(-2)^n}{(1+r)^2\dots(n+r)^2}.$$

We get one solution to the original equation by setting $r=0$.

$$y_1(x) = 1 + \sum_{n=1}^{\infty}\frac{(-2x)^n}{(n!)^2}$$

The second solution is obtained by taking the partial derivative with respect to r of

$$y = x^r\left(1 + \sum_{n=1}^{\infty}\frac{(-2x)^n}{(1+r)^2\dots(n+r)^2}\right)$$

and then setting $r=0$.

$$\frac{\partial y}{\partial r} = \log x \, x^r\left(1 + \sum_{n=1}^{\infty}\frac{(-2x)^n}{(1+r)^2\dots(n+r)^2}\right)$$

$$+x^r\left(\sum_{n=1}^{\infty}\frac{(-2x)^n}{(1+r)^2\dots(n+r)^2}\left[-\frac{2}{1+r} - \dots - \frac{2}{n+r}\right]\right)$$

$$\left.\frac{\partial y}{\partial r}\right|_{r=0} = \log x\left(1 + \sum_{n=1}^{\infty}\frac{(-2x)^n}{(n!)^2}\right) - 2\sum_{n=1}^{\infty}\frac{(-2x)^n}{(n!)^2}\left[\frac{1}{1} + \dots + \frac{1}{n}\right]$$

$$y_2(x) = y_1(x)\log x - 2\sum_{n=1}^{\infty}\frac{(-2x)^n}{(n!)^2}H(n)$$

Roots differing by an integer.

When the roots differ by an integer, the solution procedure breaks down for the smaller root r_2 when $N + r_2 = r_1$. Then we have

$$F(N + r_2)a_N \,(= F(r_1)a_N = 0.a_N) = -\sum_{l=0}^{N-1}((l + r)P_{N-l} + Q_{N-l})a_l \ .$$

In some fortunate cases (such as Bessel's equation of order $n+\frac{1}{2}$) the right hand side of this equation vanishes identically, so that this equation becomes vaccuous, and we are at liberty to choose a_N arbitrarily; this choice being equivalent to adding a multiple of the solution corresponding to $r = r_1$ to the second solution.

Usually however the only way to maintain consistency in this equation is to set $a_0 = \cdots = a_{N-1} = 0$, so that the solution which we are calculating begins $a_N(x - c)^{N+r_2} = a_N(x - c)^{r_1}$, i.e. we merely obtain a multiple of the first solution and not a second INDEPENDENT solution.

To overcome this problem there are two techniques available. One is an extension of the procedure used for the case of repeated roots. The second assumes the appropriate form for the solution and substitutes this expression into the equation. These are illustrated in the following example.

$$x^2y'' + 2xy' + xy = 0$$

The indicial equation for the equation $x^2y''+2xy'+xy = 0$ is $r(r-1)+2r+0 = 0$, $r(r + 1) = 0$, so that the indices are 0 and -1. Corresponding to the larger of the roots we can determine a series solution in the normal manner.

$$\text{If} \quad y = \sum_{n=0}^{\infty} a_n x^{n+r} = \sum_{n=0}^{\infty} a_n x^n \quad \text{when} \quad r = 0$$

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=1}^{\infty} n a_n x^{n-1} \ ; \ xy' = \sum_{n=1}^{\infty} n a_n x^n$$

$$y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} \ ; \ x^2y'' = \sum_{n=1}^{\infty} n(n-1)a_n x^n$$

$$xy = \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=1}^{\infty} a_{n-1}x^n$$

$$x^2y'' + 2xy' + xy = \sum_{n=1}^{\infty}([n(n-1) + 2n]a_n + a_{n-1})x^n$$

$$n(n+1)a_n + a_{n-1} = 0 \ ; \ a_n = -\frac{1}{n(n+1)}a_{n-1}$$

$$a_n = \frac{(-1)^n}{n!(n+1)!}a_0$$

$$y = a_0 \sum_{n=0}^{\infty} \frac{(-x)^n}{n!(n+1)!}$$

To derive the second solution, we have two alternate procedures available.

Firstly, consider $y = \sum a_n x^{n+r}$, where $a_0 = (r+1)$.

$$\text{If} \quad y = \sum_{n=0}^{\infty} a_n x^{n+r} \ ; \ y' = \sum_{n=0}^{\infty} (n+r) a_n x^{n+r-1}$$

$$y'' = \sum_{n=0}^{\infty} (n+r)(n+r-1) a_n x^{n+r-2} \ ; \ xy = \sum_{n=1}^{\infty} a_{n-1} x^{n+r}$$

$$x^2 y'' + 2xy' + xy = [r(r-1) + 2r] a_0 x^r$$

$$+ \sum_{n=1}^{\infty} ([(n+r)(n+r-1) + 2(n+r)] a_n + a_{n-1}) x^{n+r}$$

$$= r(r+1)^2 x^r + \sum_{n=1}^{\infty} ((n+r)(n+r+1) a_n + a_{n-1}) x^{n+r}$$

If we now choose the coefficients a_n to satisfy $(n+r)(n+r+1)a_n + a_{n-1} = 0$ for $n \geq 1$, $x^2 y'' + 2xy' + xy = r(r+1)^2 x^r$.

$$\text{We have} \quad (r+1)(r+2) a_1 = -a_0 = -(r+1) \ ; \ a_1 = -\frac{1}{(r+2)}$$

$$\text{and, for} \quad n \geq 2, \quad a_n = -\frac{1}{(n+r)(n+r+1)} a_{n-1} \ = \ \frac{(-1)^n}{(2+r)^2 \ldots (n+r)^2 (n+r+1)}$$

$$\text{so that} \quad y = x^r \left((r+1) + \sum_{n=1}^{\infty} \frac{(-x)^n}{(2+r)^2 \ldots (n+r)^2 (n+r+1)} \right)$$

$$\mathbb{L}(y) = r(r+1)^2 x^r \ = \ 0 \quad \text{when} \quad r = 0 \text{ or } r = -1.$$ ∎

Both these choices give multiples of the solution already found. However, we also have the possibility

$$\mathbb{L}\left(\frac{\partial y}{\partial r}\right) = [(r+1)^2 + 2r(r+1)] x^r + r(r+1)^2 x^r \log x$$

which also vanishes when $r = -1$.

$$\frac{\partial y}{\partial r} = x^r \log x \left((r+1) + \sum_{n=1}^{\infty} \frac{(-x)^n}{(2+r)^2 \ldots (n+r)^2 (n+r+1)} \right)$$

$$+ x^r \left(1 + \sum_{n=1}^{\infty} \frac{(-x)^n}{(2+r)^2 \ldots (n+r)^2 (n+r+1)} \left[-\frac{2}{(2+r)} - \cdots - \frac{2}{(n+r)} - \frac{1}{(n+r+1)} \right] \right)$$ ∎

$$\left.\frac{\partial y}{\partial r}\right|_{r=-1} \ = \ x^{-1} \log x \left(0 + \sum_{n=1}^{\infty} \frac{(-x)^n}{(n-1)! n!} \right)$$

$$+ x^{-1} \left(1 - \sum_{n=1}^{\infty} \frac{(-x)^n}{(n-1)! n!} \left[\left(\frac{1}{1} + \cdots + \frac{1}{n-1} \right) + \left(\frac{1}{1} + \cdots + \frac{1}{n} \right) \right] \right)$$

$$= \ - \ \log x \sum_{n=0}^{\infty} \frac{(-x)^n}{n!(n+1)!} \ + \ \left(x^{-1} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!(n+1)!} [H(n) + H(n+1)] \right)$$

The alternative approach is to set

$$y = \log x \sum_{n=0}^{\infty} \frac{(-x)^n}{n!(n+1)!} \; + \; \sum_{n=-1}^{\infty} b_n x^n .$$

$$y' = \log x \; y_1'(x) - \sum_{n=0}^{\infty} \frac{(-x)^{n-1}}{n!(n+1)!} \; + \; \sum_{n=-1}^{\infty} n b_n x^{n-1}$$

$$y'' = \log x \; y_1''(x) + \frac{1}{x} \sum_{n=0}^{\infty} (-n) \frac{(-x)^{n-1}}{n!(n+1)!}$$

$$+ \; \sum_{n=0}^{\infty} (n-1) \frac{(-x)^{n-2}}{n!(n+1)!} \; + \; \sum_{n=-1}^{\infty} n(n-1) b_n x^{n-2}$$

$$xy = \log x \, x y_1(x) + \sum_{n=0}^{\infty} b_{n-1} x^n$$

$$xy' = \log x \, x y_1'(x) + \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+1)!} x^n \; + \; \sum_{n=-1}^{\infty} n b_n x^n$$

$$x^2 y'' = \log x \, x^2 y_1''(x) + \sum_{n=0}^{\infty} n \frac{(-1)^n}{n!(n+1)!} x^n$$

$$+ \; \sum_{n=0}^{\infty} (n-1) \frac{(-1)^n}{n!(n+1)!} x^n \; + \; \sum_{n=-1}^{\infty} n(n-1) b_n x^n$$

$$x^2 y'' + 2xy' + xy \;=\; \log x \, [x^2 y_1'' + 2x y_1' + x y_1]$$

$$+ \; \sum_{n=0}^{\infty} [n + (n-1) + 2] \frac{(-1)^n}{n!(n+1)!} x^n \; + \; \sum_{n=-1}^{\infty} [n(n-1) + 2n] b_n x^n \; + \; \sum_{n=0}^{\infty} b_{n-1} x^n$$

$$= \; \sum_{n=0}^{\infty} (n + (n+1)) \frac{(-1)^n}{n!(n+1)!} x^n \; + \; \sum_{n=0}^{\infty} [n(n+1) b_n + b_{n-1}] x^n$$

Setting $\quad b_n = \dfrac{(-1)^n}{n!(n+1)!} c_n \quad$ for $\quad n \geq 0$

we have $\quad 1 + b_{-1} = 0$

$$(n + (n+1)) \frac{(-1)^n}{n!(n+1)!} + n(n+1) \frac{(-1)^n}{n!(n+1)!} c_n + \frac{(-1)^{n-1}}{(n-1)!n!} c_{n-1} = 0$$

$$c_n = c_{n-1} - \frac{1}{n} - \frac{1}{n+1}$$

Taking $\quad c_0 = -1 \; , \; c_n = -H(n) - H(n+1) \; ,$ and

$$y = \log x \, y_1(x) - \left(x^{-1} + \sum_{n=0}^{\infty} \frac{(-x)^n}{n!(n+1)!} [H(n) + H(n+1)] \right) .$$

Bessel's equation and Bessel Functions.

Bessel's equation is

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0 \ .$$

It originally arose in the solution of astronomical problems associated with orbital perturbations, but it also appears in cylindrical problems.

The solutions are usually presented in standardised form as $J_\nu(x)$ and $Y_\nu(x)$, known as the *Bessel Functions of order* ν *of the First and Second Kinds*.

Using the method of Frobenius we make the substitution

$$y = \sum_{k=0}^{\infty} a_k x^{k+r} \ .$$

This gives

$$\sum_{k=0}^{\infty} \left((k+r)(k+r-1) + (k+r) - \nu^2\right) a_k x^{k+r} = -\sum_{k=2}^{\infty} a_{k-2} x^{k+r} \ .$$

For $k = 0$ we obtain

$$(r^2 - \nu^2)a_0 = 0 \ ,$$

so that the roots of the indicial equation are ν and $-\nu$, which for the moment we will assume do not differ by an integer.

For $k = 1$ we obtain

$$((r+1)^2 - \nu^2)a_1 = (1 \pm 2\nu)a_1 = 0 \ .$$

Since we are assuming 2ν is not an integer, this gives $a_1 = 0$.

For $k \geq 2$ we have the recurrence relation

$$((k+r)^2 - \nu^2)a_k = -a_{k-2} \ .$$

Since $a_1 = 0$, it follows that $a_{2m+1} = 0$ for all m, so that we need only consider $k = 2m$.

$$\begin{aligned}
(4m^2 \pm 4m\nu)a_{2m} &= -a_{2m-2} \\
4m(m \pm \nu)a_{2m} &= -a_{2m-1} \\
a_{2m} &= \frac{-1}{4}\frac{1}{m}\frac{1}{m \pm \nu}a_{2(m-1)} \\
a_2 &= \frac{-1}{2^2}\frac{1}{1}\frac{1}{1 \pm \nu}a_0 \\
&= \frac{-1}{2^2}\frac{1}{1}\frac{\Gamma(1 \pm \nu)}{\Gamma(2 \pm \nu)}a_0 \\
a_4 &= \frac{-1}{2^2}\frac{1}{2}\frac{1}{2 \pm \nu}a_2 \\
&= \frac{(-1)^2}{2^4}\frac{1}{2!}\frac{\Gamma(1 \pm \nu)}{\Gamma(3 \pm \nu)}a_0 \\
a_{2m} &= \frac{(-1)^m}{2^{2m}}\frac{1}{m!}\frac{\Gamma(1 \pm \nu)}{\Gamma(m+1 \pm \nu)}a_0 \\
&= \frac{(-1)^m}{2^{2m\pm\nu}}\frac{1}{m!}\frac{1}{\Gamma(m+1 \pm \nu)} \\
\text{if} \quad a_0 &= \frac{1}{2^{\pm\nu}\Gamma(1 \pm \nu)}
\end{aligned}$$

Using these coefficients we have the Bessel Functions of the First Kind

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+1+\nu)} \left(\frac{x}{2}\right)^{2m+\nu}$$
$$J_{-\nu}(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\Gamma(m+1-\nu)} \left(\frac{x}{2}\right)^{2m-\nu}$$

Returning to the original equation, we make the substitution $y = x^{-1/2}w$.

$$y' = -\frac{1}{2}x^{-3/2}w + x^{-1/2}w' \ ; \ y'' = \frac{3}{4}x^{-5/2}w - x^{-3/2}w' + x^{-1/2}w''$$
$$x^2y'' + xy' + (x^2 - \nu^2)y =$$
$$= x^{3/2}w'' - x^{1/2}w' + \frac{3}{4}x^{-1/2}w - \frac{1}{2}x^{-1/2}w + x^{1/2}w' + x^{3/2}w - \nu^2x^{-1/2}w$$
$$\text{Therefore} \quad x^{3/2}\left(w'' + w + \frac{1}{x^2}(\frac{1}{4} - \nu^2)w\right) = 0$$

For large values of x this leads to the approximation

$$w'' + w = 0 \ ,$$

so that $w \simeq a\cos(x+b)$ (for some a and b) for large values of x. Hence we expect the Bessel functions to oscillate with algebraically decreasing amplitude as $x \to \infty$. A more thorough analysis (**MA373**) gives the formulae

$$J_\nu(x) \simeq \sqrt{\frac{2}{\pi x}} \cos\left(x - \left(\frac{\nu}{2} + \frac{1}{4}\right)\pi\right)$$
$$J_{-\nu}(x) \simeq \sqrt{\frac{2}{\pi x}} \cos\left(x - \left(-\frac{\nu}{2} + \frac{1}{4}\right)\pi\right)$$

The Bessel Function of the Second Kind is defined in terms of this asymptotic behaviour. If we take

$$Y_\nu(x) = \frac{1}{\sin\nu\pi}\left(\cos\nu\pi\, J_\nu(x) - J_{-\nu}(x)\right) \ ,$$

and use the asymptotic formulae above, we obtain

$$Y_\nu(x) \simeq \sqrt{\frac{2}{\pi x}} \sin\left(x - \left(\frac{\nu}{2} + \frac{1}{4}\right)\pi\right) \ .$$

The defining formula for $Y_\nu(x)$ only holds for non-integral values of ν; for integral values we take the limit.

$$Y_n(x) = \lim_{\nu\to n} \frac{1}{\sin\nu\pi}\left(\cos\nu\pi\, J_\nu(x) - J_{-\nu}(x)\right) \ ,$$

Bessel Functions of Order $n + \frac{1}{2}$.

When $\nu = n + \frac{1}{2}$, $2\nu = 2n + 1$, and the roots of the indicial equation differ by an integer. This means that the solution process may run into difficulties. However, the recurrence equation in which the difficulty arises is

$$(2n + 1)(2n + 1 - 2\nu)a_{2n+1} = -a_{2n-1} \ ,$$

and our solution procedure has already set $a_k = 0$ for all odd k. Hence, this equation is always consistent, and we recover the above solution form for $J_{-\nu}(x)$ if we set $a_{2n+1} = 0$.

Note also that the equation $w'' + w = 0$ for w becomes exact if $\nu = \frac{1}{2}$, so that we have

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos(x - \frac{\pi}{2}) = \sqrt{\frac{2}{\pi x}} \sin x$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x \ .$$

Bessel Functions of Integer Order.

Firstly, let us consider $J_n(x)$.

For integral values, $\Gamma(N + 1) = N!$, so that the expansion of $J_n(x)$ for $n \geq 0$ is

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m + n)!} \left(\frac{x}{2}\right)^{2m+n} \ .$$

This power series converges for all values of x, and has no fractional powers of x, so that the function $J_n(x)$ is *entire*.

(Other entire functions with which you are familiar are e^x, $\sin x$, and $\cos x$.)

Although they appear forbidding, there is a simple program for calculating the values of $J_n(x)$ on a pocket calculator, and these functions can be considered as *elementary functions*.

When we consider $J_{-n}(x)$, the situation degenerates. In contrast to the case above, if $\nu = n$, the recurrence relation in which trouble occurs is

$$2n(2n - 2\nu)a_{2n} = -a_{2n-2} \ ,$$

so that we have to set $a_0 = a_2 = \cdots = a_{2n-2} = 0$. With appropriate scaling this gives the result

$$J_{-n}(x) = (-1)^n J_n(x)$$

so that this is not an independent solution. However the limiting process for Y_n does produce an independent solution in the form

$$Y_n(x) = \frac{2}{\pi} \left[\log\left(\frac{x}{2}\right) + \gamma\right] J_n(x) - \frac{1}{\pi} \sum_{k=0}^{n-1} (n - k - 1)! \left(\frac{x}{2}\right)^{2k-n}$$

$$- \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m + n)!} \left(\frac{x}{2}\right)^{2m+n} [H(m) + H(m + n)]$$

where $\gamma = .5772156\ldots$ is Euler's constant, and

$$H(p) = 1 + \frac{1}{2} + \cdots + \frac{1}{p} \ , \ p > 0 \ , H(0) = 0 \ .$$

Different forms for a second independent solution can also be obtained by the methods used earlier in the general cases.

Examples of Exam Questions

1. Consider the differential equation

$$(1-x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant.

(a) Determine two linearly independent solutions in powers of x for $|x| < 1$.

(b) Show that if α is a nonnegative integer n, then there is a polynomial solution of degree n.

(c) Find a polynomial solution for each of the cases $\alpha = n = 0, 1, 2,$ and 3.

2. Find two independent solutions of the differential equation

$$y'' - 2xy' + 2py = 0$$

in the form of power series in x.

For which values of x do these series converge?

Show that if p is an integer N, the one of the solutions is a polynomial of degree N in x.

3. Use the method of Frobenius to obtain two solutions of the differential equation

$$x(1-x)y'' + (\frac{1}{2} - 3x)y' - y = 0$$

as series in powers of x.

4. Use the method of Frobenius to obtain two solutions of the differential equation

$$x(1-x)y'' + (1-2x)y' + 6y = 0$$

in a neighbourhood of $x = 0$.