THE LAPLACE TRANSFORM

The Laplace Transform is the simplest of a class of *Integral Transforms* which are used to reduce ordinary differential equations to algebraic equations, and to reduce partial differential equations to ordinary differential equations.

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Consider a function $f(t)$ which is defined for $t \geq 0$, which is piecewise continuous, and for which there is some (real) constant a such that $e^{-at}f(t) \to 0$ as $t \to \infty$. Under these conditions, the improper integral

$$\int_0^\infty e^{-st} f(t)\, dt$$

exists for all values of the parameter s such that $Re(s) > a$.

This integral, whose value depends on the parameter s, is called the **Laplace Transform** of the function $f(t)$.

e.g. If $f(t) = e^{ct}$, $e^{-at}f(t) \to 0$ for any $a > c$, and

$$\int_0^\infty e^{-st} f(t)\, dt = \int_0^\infty e^{-(s-c)t}\, dt = \frac{1}{s-c} \quad \text{provided} \quad Re(s) > c\ .$$

e.g. If $f(t) = t^{r-1}$ $(r > 0)$, $e^{-at}f(t) \to 0$ for any $a > 0$, and

$$\begin{aligned} \int_0^\infty e^{-st} f(t)\, dt &= \int_0^\infty e^{-st} t^{r-1}\, dt \\ &= \int_0^\infty e^{-\tau} \left(\frac{\tau}{s}\right)^{r-1} \frac{d\tau}{s} \\ &= \frac{1}{s^r} \int_0^\infty e^{-\tau} \tau^{r-1}\, d\tau \\ &= \frac{\Gamma(r)}{s^r} \end{aligned}$$

where the Gamma function $\Gamma(r)$ is defined by the integral

$$\Gamma(r) = \int_0^\infty e^{-t} t^{r-1}\, dt\ .$$

The Gamma function is a generalisation of the factorial function for integers. This function is defined for all $r > 0$ by the integral above. If we integrate by parts, we obtain the recurrence relation

$$\begin{aligned} \Gamma(r+1) &= \int_0^\infty e^{-t} t^r\, dt \\ &= -e^{-t}t^r|_0^\infty + \int_0^\infty e^{-t}(rt^{r-1})\, dt \\ &= r\Gamma(r)\ . \end{aligned}$$

In particular, when $r = 1$,

$$\Gamma(1) = \int_0^\infty e^{-t}\, dt = -e^{-t}|_0^\infty = 1\ ,$$

so that for integer n, $\Gamma(n+1) = n!$.

Applying this last result to the function $f(t) = t^n$ for integral $n \geq 0$, we obtain the Laplace transform of f as $n!/s^{n+1}$.

Notation.

For convenience, let us use lower case letters to denote our pre-transformed functions of t and use the corresponding upper case letter to denote its Laplace transform as a function of s. We will also use the notation $\mathcal{L}(f(t))$ to denote the Laplace transform. Hence

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)\, dt\ .$$

Elementary properties of the Laplace Transform.

1. The Laplace transform is a *linear operator*.

$$\begin{aligned}\mathcal{L}(af_1(t) + bf_2(t)) &= \int_0^\infty e^{-st}(af_1(t) + bf_2(t))\, dt \\ &= a\int_0^\infty e^{-st} f_1(t)\, dt + b\int_0^\infty e^{-st} f_2(t)\, dt \\ &= a\mathcal{L}(f_1(t)) + b\mathcal{L}(f_2(t))\end{aligned}$$

e.g. Find $\mathcal{L}(\cosh at)$.

$$\begin{aligned}\mathcal{L}(\cosh at) &= \mathcal{L}(\frac{1}{2}e^{at} + \frac{1}{2}e^{-at}) \\ &= \frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}(e^{-at}) \\ &= \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a} \\ &= \frac{1}{2}\frac{(s+a)+(s-a)}{(s-a)(s+a)} \\ &= \frac{s}{s^2-a^2}\end{aligned}$$

e.g Find $\mathcal{L}(t^2 + 3t + 2)$.

$$\begin{aligned}\mathcal{L}(t^2 + 3t + 2) &= \mathcal{L}(t^2) + 3\mathcal{L}(t^1) + 2\mathcal{L}(t^0) \\ &= \frac{2}{s^3} + \frac{3}{s^2} + \frac{2}{s}\end{aligned}$$

e.g. Find $\mathcal{L}(\sin bt)$.

$$\begin{aligned}\mathcal{L}(\sin bt) &= \mathcal{L}(\frac{1}{2i}e^{ibt} - \frac{1}{2i}e^{-ibt}) \\ &= \frac{1}{2i}\mathcal{L}(e^{ibt}) - \frac{1}{2i}\mathcal{L}(e^{-ibt}) \\ &= \frac{1}{2i}\frac{1}{s-ib} - \frac{1}{2i}\frac{1}{s+ib} \\ &= \frac{1}{2i}\frac{(s+ib)-(s-ib)}{(s-ib)(s+ib)} \\ &= \frac{b}{s^2+b^2}\end{aligned}$$

2. The second elementary property of the Laplace Transform is the one which makes it so useful.

If the function $f(t)$ is continuous, and its derivative has a Laplace Transform, then

$$\begin{aligned} \mathcal{L}\left(\frac{df}{dt}\right) &= \int_0^\infty e^{-st}\frac{df}{dt}\,dt \\ &= e^{-st}f(t)|_0^\infty + s\int_0^\infty e^{-st}f(t)\,dt \\ &= -f(0) + s\mathcal{L}(f(t)) \ . \end{aligned}$$

If we apply this formula to the second derivative, we obtain

$$\begin{aligned} \mathcal{L}\left(\frac{d^2f}{dt^2}\right) &= s\mathcal{L}\left(\frac{df}{dt}\right) - \frac{df}{dt}(0) \\ &= s^2\mathcal{L}(f(t)) - sf(0) - \dot{f}(0) \ , \end{aligned}$$

and proceeding inductively,

$$\mathcal{L}\left(\frac{d^nf}{dt^n}\right) = s^nF(s) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - \cdots - \frac{d^{n-1}f}{dt^{n-1}}(0) \ .$$

e.g. Use the Laplace Transform to solve the initial value problem

$$\ddot{y} - \dot{y} - 6y = 0 \ ; \quad y(0) = 1, \quad \dot{y}(0) = -1 \ .$$

Let $Y(s) = \mathcal{L}(y(t))$. then

$$\begin{aligned} \mathcal{L}(\ddot{y} - \dot{y} - 6y) &= \mathcal{L}(\ddot{y}) - \mathcal{L}(\dot{y}) - 6\mathcal{L}(y) \\ &= s^2Y(s) - s + 1 - (sY(s) - 1) - 6Y(s) \\ &= (s^2 - s - 6)Y(s) - s + 2 \end{aligned}$$

Since $\mathcal{L}(0) = 0$, we must have

$$\begin{aligned} Y(s) &= \frac{s-2}{s^2 - s - 6} \\ &= \frac{s-2}{(s-3)(s+2)} \\ &= \frac{1/5}{s-3} + \frac{4/5}{s+2} \\ &= \frac{1}{5}\mathcal{L}(e^{3t}) + \frac{4}{5}\mathcal{L}(e^{-2t}) \\ &= \mathcal{L}\left(\frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}\right) \\ y(t) &= \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t} \end{aligned}$$

e.g. Use the Laplace Transform to solve the initial value problem

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} \ ; \quad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \ .$$

Let $X(s) = \mathcal{L}(x(t))$ and $Y(s) = \mathcal{L}(y(t)$. Then, applying the Laplace Transform to the system, and equating both sides, we obtain

$$\begin{aligned}
sX(s) - a &= 3X + Y \\
sY(s) - b &= 2X + 2Y \\
\begin{pmatrix} (s-3) & -1 \\ -2 & (s-2) \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} &= \begin{pmatrix} a \\ b \end{pmatrix} \\
\begin{pmatrix} X \\ Y \end{pmatrix} &= \begin{pmatrix} (s-3) & -1 \\ -2 & (s-2) \end{pmatrix}^{-1} \begin{pmatrix} a \\ b \end{pmatrix}
\end{aligned}$$

The determinant of the matrix is $(s-3)(s-2)-2 = s^2-5s+4 = (s-4)(s-1)$, so that the inverse is

$$\begin{pmatrix} (s-3) & -1 \\ -2 & (s-2) \end{pmatrix}^{-1} = \frac{1}{(s-4)(s-1)} \begin{pmatrix} (s-2) & 1 \\ 2 & (s-3) \end{pmatrix} .$$

Substituting this into the equation for X and Y, we obtain

$$\begin{aligned}
X(s) &= \frac{a(s-2)+b}{(s-4)(s-1)} \\
&= \frac{2a+b}{3}\frac{1}{s-4} + \frac{a-b}{3}\frac{1}{s-1} \\
&= \frac{2a+b}{3}\mathcal{L}(e^{4t}) + \frac{a-b}{3}\mathcal{L}(e^{t}) \\
x(t) &= \frac{2a+b}{3}e^{4t} + \frac{a-b}{3}e^{t} \\
Y(s) &= \frac{2a+b(s-3)}{(s-4)(s-1)} \\
&= \frac{2a+b}{3}\frac{1}{s-4} + \frac{2b-2a}{3}\frac{1}{s-1} \\
&= \frac{2a+b}{3}\mathcal{L}(e^{4t}) + \frac{2b-2a}{3}\mathcal{L}(e^{t}) \\
y(t) &= \frac{2a+b}{3}e^{4t} + \frac{2b-2a}{3}e^{t}
\end{aligned}$$

e.g.. Solve the given system of equations.

$$\begin{aligned}
x_1' &= 2x_1 - 5x_2 - \sin 2t, \quad & x_1(0) &= 0 \\
x_2' &= \ x_1 - 2x_2 + t, & x_2(0) &= 1
\end{aligned}$$

Let $X_1(s) = \mathcal{L}(x_1(t))$, and $X_2(s) = \mathcal{L}(x_2(t))$ be the Laplace transforms of the unknown functions. Then

$$\begin{aligned}
sX_1(s) - 0 &= 2X_1 - 5X_2 - \frac{2}{s^2+4} \\
sX_2(s) - 1 &= \ X_1 - 2X_2 + \frac{1}{s^2} \\
\begin{pmatrix} s-2 & 5 \\ -1 & s+2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} &= \begin{pmatrix} -2/(s^2+4) \\ 1+1/s^2 \end{pmatrix} \\
\begin{pmatrix} X_1 \\ X_2 \end{pmatrix} &= \frac{1}{s^2-4+5} \begin{pmatrix} s+2 & -5 \\ 1 & s-2 \end{pmatrix} \begin{pmatrix} -2/(s^2+4) \\ 1+1/s^2 \end{pmatrix}
\end{aligned}$$

$$\begin{aligned}
X_1 &= \frac{1}{s^2+1}\left(\frac{-2(s+2)}{s^2+4} - \frac{5(s^2+1)}{s^2}\right) \\
&= \frac{-2s-4}{(s^2+1)(s^2+4)} - \frac{5}{s^2} \\
&= \frac{2}{3}\left(\frac{s+2}{s^2+4} - \frac{s+2}{s^2+1}\right) - \frac{5}{s^2} \\
x_1(t) &= \frac{2}{3}(\cos 2t + \sin 2t - \cos t - 2\sin t) - 5t
\end{aligned}$$

$$\begin{aligned}
X_2 &= \frac{1}{s^2+1}\left(-\frac{2}{s^2+4} + \frac{(s-2)(s^2+1)}{s^2}\right) \\
&= -\frac{2}{(s^2+1)(s^2+4)} + \frac{s-2}{s^2} \\
&= \frac{2}{3}\left(\frac{1}{s^2+4} - \frac{1}{s^2+1}\right) + \left(\frac{1}{s} - \frac{2}{s^2}\right) \\
x_2(t) &= \frac{1}{3}\sin 2t - \frac{2}{3}\sin t + 1 - 2t
\end{aligned}$$

The First Shift Theorem.

Suppose $F(s) = \mathcal{L}(f(t))$. Then

$$\begin{aligned}
\mathcal{L}(e^{at}f(t)) &= \int_0^\infty e^{-st}e^{at}f(t)\,dt \\
&= \int_0^\infty e^{-(s-a)t}f(t)\,dt \\
&= F(s-a)
\end{aligned}$$

This result is known as the *First Shift Theorem.* It enables us to increase simply the number of functions for which we know the Laplace Transform.

e.g.

$$\begin{aligned}
\mathcal{L}(te^{at}) &= \frac{1}{(s-a)^2} \\
\mathcal{L}(t^n e^{at}) &= \frac{n!}{(s-a)^{n+1}} \\
\mathcal{L}(e^{at}\cos bt) &= \frac{(s-a)}{(s-a)^2+b^2} \\
\mathcal{L}(e^{at}\sin bt) &= \frac{b}{(s-a)^2+b^2}
\end{aligned}$$

e.g. Use the Laplace Transform to solve the initial value problem

$$\ddot{x} - 2\dot{x} + 2x = e^{-t}\ ; \quad x(0) = 0, \quad \dot{x}(0) = 1\ .$$

Let $X(s) = \mathcal{L}(x(t)$. Then

$$\begin{aligned}
\mathcal{L}(\ddot{x} - 2\dot{x} + 2x) &= \mathcal{L}(\ddot{x}) - 2\mathcal{L}(\dot{x}) + 2\mathcal{L}(x) \\
&= s^2X - 1 - 2sX + 2X \ = \ (s^2 - 2s + 2)X - 1 \\
\mathcal{L}(e^{-t}) &= \frac{1}{s+1} \\
(s^2 - 2s + 2)X - 1 &= \frac{1}{s+1} \\
((s-1)^2 + 1)X &= 1 + \frac{1}{s+1} \ = \ \frac{s+2}{s+1} \\
X(s) &= \frac{s+2}{(s+1)((s-1)^2+1)} \\
&= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-8}{(s-1)^2+1} \\
&= \frac{1}{5}\frac{1}{s+1} - \frac{1}{5}\frac{s-1}{(s-1)^2+1} + \frac{7}{5}\frac{1}{(s-1)^2+1} \\
&= \frac{1}{5}\mathcal{L}(e^{-t}) - \frac{1}{5}\mathcal{L}(e^t\cos t) + \frac{7}{5}\mathcal{L}(e^t\sin t) \\
x(t) &= \frac{1}{5}e^{-t} - \frac{1}{5}e^t\cos t + \frac{7}{5}e^t\sin t
\end{aligned}$$

e.g. Use the Laplace Transform to solve the initial value problem

$$\dddot{x} + 3\ddot{x} + 3\dot{x} + x = te^{-t} \ ; \quad x(0) = 1; \quad \dot{x}(0) = -1; \quad \ddot{x}(0) = 2 \ .$$

Let $X(s) = \mathcal{L}(x(t))$. Taking the Laplace Transform of both sides of the equation gives

$$\begin{aligned}
s^3X - s^2 + s - 2 + 3(s^2X - s + 1) + 3(sX - 1) + X &= \frac{1}{(s+1)^2} \\
(s^3 + 3s^2 + 3s + 1)X - s^2 - 2s - 2 &= \frac{1}{(s+1)^2} \\
(s+1)^3X &= (s+1)^2 + 1 + \frac{1}{(s+1)^2} \\
X &= \frac{1}{s+1} + \frac{1}{(s+1)^3} + \frac{1}{(s+1)^5} \\
x(t) &= e^{-t} + \frac{1}{2}t^2e^{-t} + \frac{1}{24}t^4e^{-t}
\end{aligned}$$

The second Shift Theorem.

Suppose that $\mathcal{L}(f(t) = F(s)$. What meaning can be attached to $e^{-cs}F(s)$ for $c > 0$?

$$\begin{aligned} e^{-cs}F(s) &= e^{-cs}\int_0^\infty e^{-st}f(t)\,dt \\ &= \int_0^\infty e^{-s(t+c)}f(t)\,dt \\ &= \int_c^\infty e^{-s\tau}f(\tau-c)\,d\tau \\ &= \int_0^c e^{-s\tau}0\,dt + \int_c^\infty e^{-s\tau}f(\tau-c)\,d\tau \\ &= \int_0^\infty e^{-s\tau}H(\tau-c)f(\tau-c)\,d\tau \\ &= \mathcal{L}(H(t-c)f(t-c)) \end{aligned}$$

$$\text{where} \qquad H(t-c) = \begin{cases} 0 & \text{if} \quad t < c \\ 1 & \text{if} \quad t > c \end{cases}$$

(**Note:** The text book uses the notation $u_c(t)$ for $H(t-c)$.)

This result is known as the *second Shift Theorem.*

It has three principal applications:

1. Laplace Transforms of discontinuous functions:

Consider the function

$$f(t) = \begin{cases} e^t & 0 < t < 1 \\ e^{-t} & 1 < t < 2 \\ 0 & t > 2 \end{cases} .$$

We could calculate its Laplace Transform directly from the definition.

However, as an exercise we will use the second Shift Theorem (and the linearity property of the Laplace Transform).

If $f_1(t) = e^t, 0 < t < 1$, and 0 elsewhere, then $f_1(t) = e^t - H(t-1)e^{(t-1)+1}$.

Therefore $\mathcal{L}(f_1(t)) = 1/(s-1) - e^{-s}e/(s-1) = (1-e^{1-s})/(s-1)$.

If $f_2(t) = e^{-t}, 1 < t < 2$, and 0 elsewhere, then

$$f_2(t) = H(t-1)e^{-((t-1)+1)} - H(t-2)e^{-((t-2)+2)} .$$

Therefore

$$\mathcal{L}(f_2(t)) = e^{-s}e^{-1}/(s+1) - e^{-2s}e^{-2}/(s+1) = (e^{-(s+1)} - e^{-(s+2)})/(s+1) .$$

Combining these results we have

$$\mathcal{L}(f(t)) = \frac{1-e^{1-s}}{s-1} + \frac{e^{-(s+1)} - e^{-(s+2)}}{s+1} .$$

While the use of the shift theorem is optional when deriving the transforms, it is necessary when finding the inverse transforms.

e.g. Determine the function whose Laplace transform is $1/s(1+e^{-s})$.

For $s > 0$, $0 < e^{-s} < 1$, so that we can expand $1/(1+e^{-s})$ as an infinite Geometric Progression.

$$\begin{aligned}
\frac{1}{1+e^{-s}} &= 1 - e^{-s} + e^{-2s} - e^{-3s} + \ldots \\
&= \sum_{n=0}^{\infty} (-1)^n e^{-ns} \\
\mathcal{L}^{-1}\left(\frac{1}{s}\right) &= 1 \\
\mathcal{L}^{-1}\left(\frac{1}{s}e^{-ns}\right) &= H(t-n) \\
\mathcal{L}^{-1}\left(\frac{1}{s(1+e^{-s})}\right) &= \mathcal{L}^{-1}\left(\sum_{n=0}^{\infty}(-1)^n \frac{1}{s}e^{-ns}\right) \\
&= \sum_{n=0}^{\infty}(-1)^n H(t-n) \\
&= \begin{cases} 1 & 2m < t < 2m+1 \\ 0 & 2m+1 < t < 2m+2 \end{cases}
\end{aligned}$$

2. The solution of constant coefficient difference equations.
Consider the difference equation

$$a_n - a_{n-1} = n \; ; \quad a_{-1} = \alpha \; .$$

We can consider a_n as the value of some function $f(t)$ when $t = n$. This means that we can replace the difference equation on the integers by the continuum difference equation

$$f(t) - f(t-1) = t \; ; \quad f(t) = \alpha \; , \; -1 < t < 0 \; .$$

If $F(s) = \mathcal{L}(f(t))$, then

$$\begin{aligned}
\mathcal{L}(f(t-1)) &= \int_0^{\infty} e^{-st} f(t-1)\, dt \\
&= \int_{-1}^{\infty} e^{-s(\tau+1)} f(\tau)\, d\tau \\
&= e^{-s}\int_{-1}^{0} e^{-s\tau}\alpha\, d\tau + e^{-s}\int_0^{\infty} e^{-s\tau} f(\tau)\, d\tau \\
&= \frac{\alpha e^{-s}}{s}(e^s - 1) + e^{-s}F(s)
\end{aligned}$$

Applying the Laplace transform to the continuum difference equation we obtain

$$\begin{aligned}
F(s) - e^{-s}F(s) - \frac{\alpha}{s}(1-e^{-s}) &= \frac{1}{s^2} \\
(1-e^{-s})F(s) &= \frac{\alpha}{s}(1-e^{-s}) + \frac{1}{s^2} \\
F(s) &= \frac{\alpha}{s} + \frac{1}{s^2}\frac{1}{1-e^{-s}} \\
&= \frac{\alpha}{s} + \frac{1}{s^2}\sum_{k=0}^{\infty} e^{-ks} \\
&= \mathcal{L}(\alpha) + \mathcal{L}\left(\sum_{k=0}^{\infty} H(t-k)(t-k)\right) \\
f(t) &= \alpha + \sum_{k=0}^{\infty} H(t-k)(t-k) \\
a_n \;=\; f(n) &= \alpha + \sum_{k=0}^{n} H(n-k)(n-k) \\
&= \alpha + \sum_{k=0}^{n-1}(n-k) \\
&= \alpha + \frac{1}{2}n(n+1)
\end{aligned}$$

3. The solution of Differential-Difference equations.
Consider the equation

$$\dot{x}(t) + x(t-h) = 0 \ (h > 0) \ ; x(t) = 1 \ , \ t \leq 0 \ .$$

When $h = 0$, the solution for $t > 0$ is $x(t) = e^{-t}$.
However, the introduction of a delay changes the nature of the solution.

Let $X(s) = \mathcal{L}(x(t))$. Then

$$\begin{aligned}
\mathcal{L}(x(t-h)) &= \int_0^{\infty} e^{-st}x(t-h)\,dt \\
&= \int_{-h}^{\infty} e^{-s(\tau+h)}x(\tau\,d\tau \\
&= e^{-sh}\int_{-h}^{0} e^{-s\tau}\,d\tau + e^{-sh}\int_0^{\infty} e^{-s\tau}x(\tau)\,d\tau \\
&= \frac{1}{s}(1-e^{-sh}) + e^{-sh}X(s)
\end{aligned}$$

Substituting this into the Laplace transform of the equation gives

$$\begin{aligned}
sX(s) - 1 + e^{-sh}X(s) &= -\frac{1}{s}(1 - e^{-sh}) \\
X(s) &= \frac{1}{s + e^{-sh}} - \frac{1}{s}\frac{1 - e^{-sh}}{s + e^{-sh}} \\
&= \frac{s - 1 + e^{-sh}}{s(s + e^{-sh})} \\
&= \frac{1}{s} - \frac{1}{s(s + e^{-sh})} \\
\frac{1}{s + e^{-sh}} &= \frac{1}{s}\frac{1}{1 + s^{-1}e^{-sh}} \\
&= \frac{1}{s}\sum_{n=0}^{\infty}(-1)^n \frac{e^{-nhs}}{s^n} \\
&= \sum_{n=0}^{\infty}(-1)^n \frac{e^{-nhs}}{s^{n+1}} \\
X(s) &= \frac{1}{s} - \sum_{n=0}^{\infty}(-1)^n \frac{e^{-nhs}}{s^{n+2}} \\
&= \mathcal{L}(1) - \sum_{n=0}^{\infty}(-1)^n \mathcal{L}\left(H(t - nh)\frac{(t - nh)^{(n+1)}}{(n+1)!}\right) \\
x(t) &= \begin{cases} 1 - t & 0 < t < h \\ 1 - t + \frac{1}{2}(t - h)^2 & h < t < 2h \\ 1 - t + \frac{1}{2}(t - h)^2 - \frac{1}{6}(t - 2h)^3 & 2h < t < 3h \end{cases} \qquad \text{etc.}
\end{aligned}$$

Depending on the value of h, this solution can decay monotonically, decay with oscillation or even become unbounded.

The Convolution Integral.

Suppose that $\mathcal{L}(f(t)) = F(s)$, and $\mathcal{L}(g(t)) = G(s)$. Is there a function $h(t)$ such that $\mathcal{L}(h(t) = F(s)G(s)$?

$$\begin{aligned}
F(s)G(s) &= \int_0^\infty e^{-s\tau} f(\tau)\,d\tau \int_0^\infty e^{-sy} g(y)\,dy \\
&= \int_0^\infty f(\tau)\left(\int_0^\infty e^{-s(\tau+y)} g(y)\,dy\right)\,d\tau \\
&= \int_0^\infty f(\tau)\left(\int_\tau^\infty e^{-st} g(t-\tau)\,dt\right)\,d\tau \\
&= \int_0^\infty e^{-st}\left(\int_0^t f(\tau)g(t-\tau)\,d\tau\right)\,dt \\
&= \mathcal{L}\left(\int_0^t f(\tau)g(t-\tau)\,d\tau\right) \\
&= \mathcal{L}\left(\int_0^t f(t-\tau)g(t)\,d\tau\right)
\end{aligned}$$

This integral is known as the convolution of the functions f and g, and is designated by $f * g$.

Some applications of the convolution integral are illustrated in the following examples.

p313 - 8-11. Find the inverse Laplace transform of each of the following functions by using the convolution theorem.

(8) $$F(s) = \frac{1}{s^4(s^2+1)}$$

We can write $\dfrac{1}{s^4(s^2+1)} = \dfrac{1}{s^4}\dfrac{1}{s^2+1}$; $\mathcal{L}^{-1}\left(\dfrac{1}{s^4}\right) = \dfrac{t^3}{3!}$; $\mathcal{L}^{-1}\left(\dfrac{1}{s^2+1}\right) = \sin t$

$$\mathcal{L}^{-1}(F(s)) = \int_0^t \frac{(t-u)^3}{6}\sin u\,du = \left[\frac{(t-u)^3}{6}(-\cos u)\right]_0^t - \int_0^t \frac{(t-u)^2}{2}\cos u\,du$$

$$= \frac{t^3}{6} - \left[\frac{(t-u)^2}{2}\sin u\right]_0^t - \int_0^t (t-u)\sin u\,du = \frac{t^3}{6} - [(t-u)(-\cos u)]_0^t + \int_0^t \cos u\,du$$

$$= \frac{t^3}{6} - t + \sin t$$

$\blacksquare$

(9) $$F(s) = \frac{s}{(s+1)(s^2+4)}$$

$$\frac{s}{(s+1)(s^2+4)} = \frac{1}{s+1}\frac{s}{s^2+4} \ ; \ \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t} \ ; \ \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) = \cos 2t$$

$$\mathcal{L}^{-1}(F(s)) = \int_0^t e^{-(t-u)}\cos 2u\,du = e^{-t}\int_0^t e^u \cos 2u\,du$$

$$= e^{-t}\left[\frac{e^u}{5}(\cos 2u + 2\sin 2u)\right]_0^t = \frac{1}{5}(\cos 2t + 2\sin 2t - e^{-t})$$

(10) $$F(s) = \frac{1}{(s+1)^2(s^2+4)}$$

$$\frac{1}{(s+1)^2(s^2+4)} \;=\; \frac{1}{(s+1)^2}\,\frac{1}{s^2+4}\ ;\ \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) = te^{-t}\ ;\ \mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = \frac{\sin 2t}{2}$$

$$\begin{aligned}
\mathcal{L}^{-1}(F(s)) &= \int_0^t (t-u)e^{-(t-u)}\frac{\sin 2u}{2}\,du \;=\frac{1}{2}e^{-t}\int_0^t (t-u)\,e^u \sin 2u\,du \\
&= \frac{1}{2}e^{-t}\left[(t-u)\frac{e^u}{5}(\sin 2u - 2\cos 2u)\right]_0^t + \frac{1}{10}e^{-t}\int_0^t e^u(\sin 2u - 2\cos 2u)\,du \\
&= \frac{1}{5}te^{-t} + \frac{1}{10}e^{-t}\left[\frac{e^u}{5}(\sin 2u - 2\cos 2u - 2\cos 2u - 4\sin 2u)\right]_0^t \\
&= \frac{1}{5}te^{-t} + \frac{1}{50}\left(-3\sin 2t - 4\cos 2t + 4e^{-t}\right)
\end{aligned}$$

$\blacksquare$

(11) $$F(s) = \frac{G(s)}{s^2+1}$$

$$\text{Since }\ \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t\ ;\ \mathcal{L}^{-1}(F(s)) = \int_0^t \sin(t-u)\,g(u)\,du.$$

p313 - 20. Consider the equation

$$\phi(t) + \int_0^t k(t-\xi)\phi(\xi)\,d\xi \;=\; f(t),$$

in which f and k are known functions, and ϕ is to be determined. Since the unknown function ϕ appears under an integral sign, the given equation is called an *integral equation*; in particular, it belongs to a class of integral equations known as Volterra (1860-1940) integral equations. Take the Laplace transform of the given integral equation, and obtain an expression for $\mathcal{L}(\phi(t))$ in terms of the transforms $\mathcal{L}(f(t))$ and $\mathcal{L}(k(t))$ of the given functions f and k. The inverse transform of $\mathcal{L}(\phi(t))$ is the solution of the original integral equation.

$$\text{Let}\quad \Phi(s) = \mathcal{L}(\phi(t))\ ,\ K(s) = \mathcal{L}(k(t))\ ,\ \text{and}\ \ F(s) = \mathcal{L}(f(t)).$$
$$\text{Then}\quad \Phi(s) + K(s)\Phi(s) = F(s)\ ;\ \Phi(s) = F(s)/(1+K(s)).$$

p314 - 21. Consider the Volterra integral equation

$$\phi(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi \; = \sin 2t.$$

(a) Show that if u is a function such that $u''(t) = \phi(t)$, then

$$u''(t) + u(t) - tu'(0) - u(0) = \sin 2t.$$

(b) Show that the given integral equation is equivalent to the initial value problem

$$u''(t) + u(t) = \sin 2t; \quad u(0) = 0, \quad u'(0) = 0.$$

(c) Solve the given integral equation by using the Laplace transform.

(d) Solve the initial value problem of part (b) and verify that the solution is the same as that obtained in (c).

(a) If $\phi(t) = u''(t)$, $\displaystyle\int_0^t (t-\xi)\phi(\xi)\,d\xi \; = \; \int_0^t (t-\xi)u''(\xi)\,d\xi$

$$= [(t-\xi)u'(\xi)]_0^t + \int_0^t u'(\xi)\,d\xi \; = \; -tu'(0) + u(t) - u(0).$$

$$\phi(t) + \int_0^t (t-\xi)\phi(\xi)\,d\xi \; = \; u''(t) + u(t) - tu'(0) - u(0) = \sin 2t.$$

(b) Suppose that $U(t)$ satisfies $U''(t) = \phi(t)$. Then $u(t) = U(t) - tU'(0) - U(0)$ also satisfies $u''(t) = \phi(t)$ as well as the initial conditions $u(0) = 0$, $u'(0) = 0$. Hence, in part (a) we may assume without loss of generality that $u(0) = u'(0) = 0$, from which the given result follows.

(c) Let $\mathcal{L}(\phi(t)) = \Phi(s)$. $\mathcal{L}(t) = \dfrac{1}{s^2}$; $\mathcal{L}(\sin 2t) = \dfrac{2}{s^2+4}$

$$\Phi(s) + \frac{1}{s^2}\Phi(s) = \frac{2}{s^2+4} \; ; \; \Phi(s) = \frac{2s^2}{(s^2+1)(s^2+4)} = \frac{-2/3}{s^2+1} + \frac{8/3}{s^2+4}$$

$$\phi(t) = -\frac{2}{3}\sin t + \frac{4}{3}\sin 2t$$

(d) Let $\mathcal{L}(u(t)) = U(s)$. $\mathcal{L}(u''(t)) = s^2U(s) - su(0) - u'(0) = s^2U(s)$

$$s^2U(s) + U(s) = \frac{2}{s^2+4} \; ; \; U(s) = \frac{2}{(s^2+1)(s^2+4)} \; = \; \frac{2/3}{s^2+1} - \frac{2/3}{s^2+4}$$

$$u(t) = \frac{2}{3}\sin t - \frac{1}{3}\sin 2t \; ; \; \phi(t) = u''(t) = -\frac{2}{3}\sin t + \frac{4}{3}\sin 2t.$$

Consider the general initial value problem with homogeneous initial data:

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \cdots + a_1 \frac{dx}{dt} + a_0 x = f(t)$$
$$x(0) = \frac{dx}{dt}(0) = \cdots = \frac{d^{n-1} x}{dt^{n-1}}(0) = 0$$

If $\mathcal{L}(x(t)) = X(s)$, and $\mathcal{L}(f(t)) = F(s)$, then

$$\left(a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0\right) X(s) = F(s)$$
$$p(s)X(s) = F(s) \ ; \quad X(s) = \frac{1}{p(s)} F(s) = G(s)F(s)$$
$$x(t) = \int_0^t g(t-\tau) f(\tau)\, d\tau$$

The function $g(t)$, whose transform is $G(s)$, is called the *Green's function* for the problem.

Note that the Green's function can also be found by solving the reduced equation with the initial data

$$x(0) = \frac{dx}{dt}(0) = \cdots = \frac{d^{n-2} x}{dt^{n-2}}(0) = 0 \ ; \ \frac{d^{n-1} x}{dt^{n-1}}(0) = \frac{1}{a_n} \ .$$

EXAMPLES OF EXAM QUESTIONS

1. Use the Laplace Transform to find the general solution of the system

$$\begin{aligned} \frac{dx}{dt} &= 3x - 2y \ , \\ \frac{dy}{dt} &= 2x - 2y \ . \end{aligned}$$

2. Use the Laplace Transform to solve the initial value problem

$$\ddddot{y} - 16y = 30 \sin t \ ; \ y(0) = 0; \dot{y}(0) = 2; \ddot{y}(0) = 0; \dddot{y}(0) = -14 \ .$$

3. Solve the differential equation

$$\begin{aligned} &\ddot{x} + 4x = f(t) \ ; \ x(0) = 0; \dot{x}(0) = 1 \ ; \\ &\text{where} \qquad f(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & 1 < t < \infty \end{cases} \ . \end{aligned}$$

4. Solve the integral equation

$$y(t) = t^2 + \int_0^t y(\tau) \sin(t - \tau) \, d\tau \ .$$

5. Solve the differential difference equation

$$\dot{y}(t) + y(t-1) = t^2 \ , \quad \text{if} \quad y(t) = 0 \ \text{ for } \ t \le 0 \ .$$