DYNAMICAL SYSTEMS

In this section we are interested in *autonomous* systems of differential equations. We will assume that the conditions for the existence and uniqueness of solutions hold.

The system $\dot{\underline{x}} = \underline{f}(t, \underline{x})$ is autonomous if it does not depend explicitly on the independent variable t.

For example, the system

$$\begin{aligned} \dot{x}_1 &= x_2^2 - 1 \\ \dot{x}_2 &= x_2 x_3 - x_1 \\ \dot{x}_3 &= x_3^2 - 2x_2 - 2 \end{aligned}$$

is autonomous, while the system

$$\begin{aligned} \dot{x} &= ty - t^2 \\ \dot{y} &= \ -tx + 1 \end{aligned}$$

is not.

Properties of autonomous systems.

1. Consider the autonomous system $\dot{\underline{x}} = \underline{f}(\underline{x})$. If $\underline{a}$ is a vector such that $\underline{f}(\underline{a}) = \underline{0}$, then $\underline{x} \equiv \underline{a}$ is a solution of the system.

For example, consider the autonomous system given above. The right hand side vanishes if

$$\begin{gathered} x_2^2 - 1 = 0 \ ; \quad x_2 = \pm 1 \\ x_3^2 - 2x_2 - 2 = 0 \ ; x_3^2 = 4 \ ; \ x_3 = \pm 2 \ ; \ \text{or } x_3^2 = 0 \ ; \ x_3 = 0 \\ x_2 x_3 - x_1 = 0 \ ; \ x_1 = x_2 x_3 = 2 \text{ or } \ -2 \text{ or } 0 \end{gathered}$$

The constant solutions are

$$(x_1, x_2, x_3)' = \begin{cases} (2, 1, 2)' \\ (-2, 1, -2)' \\ (0, -1, 0)' \end{cases}$$

These constant solutions are referred to as *equilibrium points* or *singular points* or *stationary points* of the autonomous system.

2. If $\underline{\phi}(t)$ is a solution of the autonomous system, so is $\underline{\phi}(t - \tau)$ for any choice of τ.

Let $\underline{\psi}(t) = \underline{\phi}(t - \tau)$. Then

$$\dot{\underline{\psi}}(t) = \dot{\underline{\phi}}(t - \tau) = \underline{f}(\underline{\phi}(t - \tau)) = \underline{f}(\underline{\psi}(t)) \ .$$

For example, the autonomous system given above has the solution

$$\underline{\phi}(t) = (8t/(1 + 4t^2), (3 - 4t^2)/(1 + 4t^2), -8t/(1 + 4t^2))' \ .$$

Replacing t by $t - \frac{1}{2}$, we obtain the solution

$$\underline{\psi}(t) = ((8t-4)/(2-4t+4t^2), (2+4t-4t^2)/(2-4t+4t^2), (4-8t)/(2-4t+4t^2))' \ .$$

This means that if we represent the solution graphically in $\mathbb{R}^n$, the curve whose parametric representation is $\underline{x} = \underline{\phi}(t)$ corresponds to a whole family of solutions of the system. Graphical portraits of this type are called *phase portraits*, and the individual solution curves are called *trajectories*. We will be particularly interested in two-dimensional phase portraits in the *phase plane*.

3. If $\underline{\phi}(t)$ is a solution of the autonomous system, and $\underline{\phi}(a) = \underline{\phi}(b)$ for some $a < b$, then either $\underline{\phi}(t)$ is an equilibrium point, or there is T such that $b = a + nT$ for some integer n and $\underline{\phi}(t+T) = \underline{\phi}(t)$ for all t. i.e. $\underline{\phi}$ is a periodic solution with period T.

Proof: If $\underline{\phi}(t)$ is an equilibrium point then $\underline{\phi}(a) = \underline{\phi}(b)$ for all a and b.

Otherwise, suppose that $\underline{\phi}$ is not a constant, and let $\underline{\psi}(t) = \underline{\phi}(t - (b-a))$. Then $\underline{\psi}(t)$ is also a solution of the autonomous system. Furthermore,

$$\underline{\psi}(b) = \underline{\phi}(b - (b-a)) = \underline{\phi}(a) = \underline{\phi}(b) \ ,$$

therefore, by the uniqueness property of the solutions, $\underline{\psi}(t) = \underline{\phi}(t)$.

Now, consider the set of all values $\tau > 0$ such that $\underline{\phi}(a+\tau) = \underline{\phi}(a)$. This set is non-empty and bounded below, therefore it has an greatest lower bound $T > 0$ such that $\underline{\phi}(a+T) = \underline{\phi}(a)$. Note that this means that $\underline{\phi}(t) = \underline{\phi}(t+T) = \underline{\phi}(t+nT)$ for all t and for all integers n.

If $b \neq a + nT$ for some integer n, then $b = a + (N+r)T$, where $0 < r < 1$, and

$$\underline{\phi}(b) = \underline{\phi}((a+rT) + NT) = \underline{\phi}(a+rT) = \underline{\phi}(a)$$

which contradicts the condition that T is the least period.

The phase line.

Firstly, we will consider the one-dimensional case; $\dot{x} = f(x)$. We assume that f and f' are continuous on $\mathbb{R}$, so that the solution exists and is unique for every choice of initial values.

The singular points of the equation are the zeros of f.

If x_1 and x_2 are consecutive zeros of f, then, by continuity, f is either positive for all $x_1 < x < x_2$, or f is negative throughout this interval.

Suppose that the first is true.

If $a \in (x_1, x_2)$, then the solution $\phi(t)$ which satisfies $\phi(0) = a$ is increasing in some neighbourhood of $t = 0$ since $\dot{\phi}(0) = f(a) > 0$. The solution will continue to increase until $\dot{\phi}(T) = 0$. But this means that $f(\phi(T)) = 0$, so that $\phi(T) = x_2$. Since the solutions are unique, the solution with $x(T) = x_2$ is $x \equiv x_2$, and therefore $\phi(t)$ is monotonic increasing for all $t > 0$, and is bounded above by x_2. Therefore the solution tends to a limit as $t \to \infty$. This means that the derivative tends to 0 as $t \to \infty$, so that $\phi \to x_2$.

Similarly, $\phi(t)$ is monotonic for $t < 0$, and $\phi(t) \to x_1$ as $t \to -\infty$.

The choice of a is arbitrary, so that this behaviour is typical of all solutions taking values between x_1 and x_2.

For any choice $0 < \epsilon \ (< x_2 - x_1)$, every solution is eventually greater than $x_1 + \epsilon$, and also is eventually between $x_2 - \epsilon$ and x_2. We say that the stationary point x_1 is unstable and that the stationary point x_2 is stable.

If $f < 0$ on the interval then the situation is reversed, and the solutions leave the neighbourhood of the unstable point x_2 and approach the stable point x_1 as $t \to \infty$.

In general, a zero x_i of f is a stable point if $f > 0$ for $x < x_i$ and $f < 0$ for $x > x_i$ in a neighbourhood of x_i. A sufficient (but not necessary) condition for this is $f'(x_i) < 0$. Similarly, if $f(x_i) = 0$ and $f'(x_i) > 0$, then x_i is an unstable point of the equation. The cases in which $f'(x_i) = 0$ have to be considered separately.

If x_0 is the first zero (in magnitude) of $f(x)$, then on the interval $(-\infty, x_0)$ we have solutions which tend to ∞ either as $t \to -\infty$ if x_0 is stable, or as $t \to \infty$ if x_0 is unstable, with similar behaviour on the interval (x_n, ∞) where x_n is the last zero of f. Hence, in the one-dimensional case trajectories either originate (at $t = -\infty$) at a singular point or terminate at a singular point (or both) or have a range $(-\infty, \infty)$ if there are no singular points.

A first order autonomous system has no non-trivial periodic solutions.

Local linearization.

An alternative non-graphical approach to the stability of the equilibrium points is to use Taylor's Theorem.

Suppose that $f(x_i) = 0$, and let $x = x_i + \xi$. Then

$$\begin{aligned} f(x) = f(x_i) + \xi f'(x_i) + \frac{1}{2}\xi^2 f''(x_i + \theta\xi) &\simeq f'(x_i)\xi \\ \dot{x} = \dot{\xi} = f(x) &\simeq f'(x_i)\xi \\ \xi &\simeq a\exp(f'(x_i)t) \end{aligned}$$

If $f'(x_i) < 0$, $\xi \to 0$ as $t \to \infty$, and x_i is a stable point, while if $f'(x_i) > 0$, $\xi \to 0$ as $t \to -\infty$, and x_i is unstable. If $f'(x_i) = 0$, it is necessary to take more terms in the Taylor Series expansion in order to obtain a valid approximation.

The Phase Plane.

We now extend these considerations to the two dimensional case.

Consider the second order autonomous system

$$\frac{dx}{dt} = P(x, y)$$
$$\frac{dy}{dt} = Q(x, y)$$

where we again assume that P and Q and their partial derivatives are continuous, so that solutions exist and are unique.

Consider the solution $x(t)$, $x(0) = a$, $y(t)$, $y(0) = b$ and the corresponding curve in the $x - y$ plane - the *phase plane*.

Provided $P(a, b)$ and $Q(a, b)$ are not both zero, the slope of this trajectory at the point (a, b) is $\tan\alpha$

$$\begin{aligned} \text{where} \qquad & \cos\alpha = P(a,b)/\sqrt{P^2(a,b) + Q^2(a,b)} \\ \text{and} \qquad & \sin\alpha = Q(a,b)/\sqrt{P^2(a,b) + Q^2(a,b)}\ . \end{aligned}$$

Hence we can obtain a graphical approximation to the behaviour of the trajectories by considering the *isoclines* of the system - i.e. the loci of the points of the trajectories which have the same slope. In particular, along the curve $P(x, y) = 0$ trajectories have vertical tangents, while along the curve $Q(x, y) = 0$ trajectories have horizontal tangents, and the local behaviour can be determined by considering neighbouring values of P and Q.

e.g.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= 1 - x^2 - y^2 \end{aligned}$$

The trajectories are horizontal on the circle $x^2 + y^2 = 1$. Inside the circle, $\dot{y} > 0$, and in the upper half plane $\dot{x} > 0$ also. This means that in the first quadrant trajectories have maxima on the circle, while in the second quadrant they have minima. Since $\dot{x} < 0$ in the lower half plane, the picture is reversed here.

Similarly, the trajectories are vertical on the line $y = 0$, and these represent maxima with respect to x outside the unit circle, and minima inside the unit circle.

This procedure fails to yield a slope at the points at which P and Q vanish simultaneously; i.e. at the singular points of the system. In order to study the behaviour of trajectories near the singular points, we use the two-dimensional Taylor series expansion.

Local linearization.

Let (a, b) be a singular point of the system, so that $P(a, b) = 0$ and $Q(a, b) = 0$. Then, setting $x = a + X$, $y = b + Y$, we have

$$\begin{aligned} P(x, y) &= P(a, b) + XP_x(a, b) + YP_y(a, b) + \\ &\quad \frac{1}{2}X^2P_{xx}(a, b) + XYP_{xy}(a, b) + \frac{1}{2}Y^2P_{yy}(a, b) + \ldots \\ Q(x, y) &= Q(a, b) + XQ_x(a, b) + YQ_y(a, b) + \\ &\quad \frac{1}{2}X^2Q_{xx}(a, b) + XYQ_{xy}(a, b) + \frac{1}{2}Y^2Q_{yy}(a, b) + \ldots \\ \dot{x} = \dot{X} &\simeq P_x(a, b)X + P_y(a, b)Y \\ \dot{y} = \dot{Y} &\simeq Q_x(a, b)X + Q_y(a, b)Y \end{aligned}$$

where the approximation is valid provided the Jacobian $P_xQ_y - Q_xP_y$, which is the determinant of the coefficient matrix, does not vanish at (a, b).

e.g. In the above example, the singular points are given by

$$y = 0 \; ; x^2 + y^2 = 1 \; ; x^2 = 1 ; x = \pm 1 \; .$$

Near $(x, y) = (1, 0)$, set $x = 1 + X$, $y = Y$. Then $y = Y$ and

$$1 - x^2 - y^2 = 1 - (1 + 2X + X^2) - Y^2 = -2X - X^2 - Y^2 \simeq -2X \; .$$

The approximate linear system is

$$\begin{aligned} \dot{X} &= Y \\ \dot{Y} &= \, -2X \end{aligned}$$

for which the Jacobian is $0.0 - (-2).1 = 2 \neq 0$.

Near $(x, y) = (-1, 0)$, set $x = -1 + X$, $y = Y$. Then $y = Y$ and

$$1 - x^2 - y^2 = 1 - (1 - 2X + X^2) - Y^2 = 2X - X^2 - Y^2 \simeq 2X \ .$$

The approximate linear system is

$$\begin{aligned}\dot{X} &= Y \\ \dot{Y} &= 2X\end{aligned}$$

for which the Jacobian is $0.0 - 2.1 = -2 \neq 0$.

Trajectories of linear systems.

Consider the linear system

$$\begin{aligned}\dot{x} &= a_{11}x + a_{12}y \\ \dot{y} &= a_{21}x + a_{22}y \ .\end{aligned}$$

Provided the determinant $a_{11}a_{22} - a_{21}a_{12}$ is non-zero, there is only one singular point, namely $(0, 0)$. Note also that the substitution $x = cX, y = cY$ does not change the system, so that the phase portrait is invariant under dilation; the picture far from the origin is the same as that near the origin.

In order to determine the phase portrait, consider the trajectory which passes throught (α, β) when $t = 0$.

If we take the Laplace Transform, so that $X = \mathcal{L}(x)$ and $Y = \mathcal{L}(y)$, we obtain

$$\begin{aligned}sX - \alpha &= a_{11}X + a_{12}Y \\ sY - \beta &= a_{21}X + a_{22}Y \\ \begin{pmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{pmatrix} \begin{pmatrix} X \\ Y \end{pmatrix} &= \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \\ \begin{pmatrix} X \\ Y \end{pmatrix} &= \frac{1}{\Delta} \begin{pmatrix} s - a_{22} & a_{12} \\ a_{21} & s - a_{11} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}\end{aligned}$$

$$\text{where} \qquad \Delta = \begin{vmatrix} s - a_{11} & -a_{12} \\ -a_{21} & s - a_{22} \end{vmatrix} = s^2 - (a_{11} + a_{22})s + (a_{11}a_{22} - a_{12}a_{21}) \ .$$

The form of the solution depends on the roots of the characteristic equation

$$s^2 - (a_{11} + a_{22})s + (a_{11}a_{22} - a_{12}a_{21}) = 0 \ ,$$

which are the eigenvalues of the coefficient matrix. Note that the condition that the determinant is non-zero means that neither of the eigenvalues is zero.

Roots real and unequal.

Suppose that $s^2 - (a_{11} + a_{22})s + (a_{11}a_{22} - a_{12}a_{21}) = (s - s_1)(s - s_2)$ where $s_1 \neq s_2$, $s_1 + s_2 = a_{11} + a_{22}$ and both s_1 and s_2 are real.

From the equation we have

$$\begin{aligned}
X(s) &= \frac{\alpha(s-a_{22})+\beta a_{12}}{(s-s_1)(s-s_2)} \\
&= \frac{1}{(s_1-s_2)}\left(\frac{\alpha(s_1-a_{22})+\beta a_{12}}{s-s_1} - \frac{\alpha(s_2-a_{22})+\beta a_{12}}{s-s_2}\right) \\
x(t) &= \frac{1}{(s_1-s_2)}\left((\alpha(s_1-a_{22})+\beta a_{12})e^{s_1t} - (\alpha(s_2-a_{22})+\beta a_{12})e^{s_2t}\right) \\
Y(s) &= \frac{\alpha a_{21}+\beta(s-a_{11})}{(s-s_1)(s-s_2)} \\
y(t) &= \frac{1}{(s_1-s_2)}\left((\alpha a_{21}+\beta(s_1-a_{11}))e^{s_1t} - (\alpha a_{21}+\beta(s_2-a_{11}))e^{s_2t}\right)
\end{aligned}$$

From this it follows that if both the eigenvalues are positive, then the solutions $x(t)$ and $y(t)$ tend to infinity as $t \to \infty$ and tend to the origin as $t \to -\infty$. Hence the singular point is unstable (*an unstable node*). Similarly, if both the eigenvalues are negative we have a stable singular point (*a stable node*).

If, finally, we have $s_2 < 0 < s_1$, then for almost all choices of α and β, $x(t)$ and $y(t)$ tend to infinity as $t \to \infty$, so that the singular point is unstable. It is also true that almost all solutions tend to infinity as $t \to -\infty$. This singular point is called *a saddle point.*

The exceptional solutions are called the *separatrices.*

If we choose α and β so that

$$\alpha(s_1-a_{22})+\beta a_{12} = 0\ ,$$

then

$$\begin{aligned}
\alpha a_{21}+\beta(s_1-a_{11}) &= \frac{1}{a_{12}}\left(\alpha a_{21}a_{12}+\beta a_{12}(s_1-a_{11})\right) \\
&= \frac{1}{a_{12}}\left(\alpha(a_{11}a_{22}-s_1s_2)-\alpha(s_1-a_{22})(s_1-a_{11})\right) \\
&= \frac{1}{a_{12}}\alpha(a_{11}a_{22}-s_1s_2-s_1^2+s_1(a_{11}+a_{22})-a_{11}a_{22}) \\
&= \frac{\alpha}{a_{12}}(-s_1s_2-s_1^2+s_1(s_1+s_2)) = 0\ ;
\end{aligned}$$

$$\begin{aligned}
\alpha(s_2-a_{22})+\beta a_{12} &= \alpha(s_2-a_{22})-\alpha(s_1-a_{22}) \\
&= \alpha(s_2-s_1) \\
x(t) &= \alpha e^{s_2t}\ ; \\
\alpha a_{21}+\beta(s_2-a_{11}) &= -\beta(s_1-a_{11})+\beta(s_2-a_{11}) \\
&= \beta(s_2-s_1) \\
y(t) &= \beta e^{s_2t}\ .
\end{aligned}$$

Therefore the two rays

$$(s_1-a_{22})x+a_{12}y = 0$$

are the trajectories of solutions which tend to the origin as $t \to \infty$.

Similarly, the two rays

$$(s_2 - a_{22})x + a_{12}y = 0$$

are the trajectories of solutions which tend to the origin as $t \to -\infty$.

If, instead of using the Laplace Transform, we had used the eigenvectors of the coefficient matrix to derive the solutions of the system, these solutions would have appeared as the solutions

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} - s_1 \end{pmatrix} e^{s_2 t}$$
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} - s_2 \end{pmatrix} e^{s_1 t}$$

e.g.

$$dx/dt = 3x - 2y$$
$$dy/dt = 2x - 2y$$

The matrix of the system is

$$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix} .$$

The eigenvalues of the problem are given by

$$|\lambda I - A| = 0$$
$$\begin{vmatrix} \lambda - 3 & 2 \\ -2 & \lambda + 2 \end{vmatrix} = (\lambda - 3)(\lambda + 2) + 4 = \lambda^2 - \lambda - 2 = 0$$
$$(\lambda - 2)(\lambda + 1) = 0 \ ; \ \lambda = 2, -1.$$

Since the eigenvalues are of opposite sign, the origin is a saddle point of the system, which is unstable.

The general solution of the system is

$$x(t) = \frac{1}{3}\left((4\alpha - 2\beta)e^{2t} - (\alpha - 2\beta)e^{-t}\right)$$
$$y(t) = \frac{1}{3}\left((2\alpha - \beta)e^{2t} - (2\alpha - 4\beta)e^{-t}\right)$$

Notice that if $\alpha = 2\beta$,

$$y(t) = \beta e^{2t} \ ; \ x(t) = 2\beta e^{2t} = 2y(t)$$

so that the rays $x = 2y$ are the unstable separatrices.

Similarly, if $\beta = 2\alpha$,

$$x(t) = \alpha e^{-t} \ ; \ y(t) = 2\alpha e^{-t} = 2x(t)$$

so that the rays $y = 2x$ are the stable separatrices.

For the other trajectories, note that

$$x(t) - 2y(t) = (\alpha - 2\beta)e^{-t} = c_1 e^{-t}$$
$$y(t) - 2x(t) = (\beta - 2\alpha)e^{2t} = c_2 e^{2t}$$
$$(x - 2y)^2(y - 2x) = c_1^2 c_2 = c$$

Two tangent nodes.

The calculation of the straight line trajectories does not depend on the signs of the eigenvalues. Therefore, the nodes - stable and unstable - also have trajectories

$$\begin{aligned} (a_{11} - s_1)x + a_{12}y = 0 \quad & \text{corresponding to} \quad e^{s_1 t} \\ (a_{11} - s_2)x + a_{12}y = 0 \quad & \text{corresponding to} \quad e^{s_2 t} \end{aligned}$$

or, equivalently, the simple vector solutions

$$\begin{aligned} \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} a_{12} \\ a_{22} - s_1 \end{pmatrix} e^{s_2 t} \\ \begin{pmatrix} x \\ y \end{pmatrix} &= \begin{pmatrix} a_{12} \\ a_{22} - s_2 \end{pmatrix} e^{s_1 t} \ . \end{aligned}$$

If $s_1 > s_2 > 0$, these trajectories both represent solutions which tend to the origin as $t \to -\infty$. However, for $t << 0$, $\exp(s_1 t) << \exp(s_2 t)$, so that for the remaining trajectories we have

$$\begin{pmatrix} x \\ y \end{pmatrix} \simeq \begin{pmatrix} a_{12} \\ s_2 - a_{11} \end{pmatrix} e^{s_2 t} \quad \text{as} \quad t \to -\infty \ ,$$

so that these trajectories are tangent to the trajectory

$$(a_{11} - s_2)x + a_{12}y = 0$$

as they approach the origin.

On the other hand, for $t >> 0$, $\exp(s_1 t) >> \exp(s_2 t)$, so that as $t \to \infty$ the trajectories tend to become parallel to the line

$$(a_{11} - s_1)x + a_{12}y = 0 \ .$$

If $s_1 < s_2 < 0$, the same picture is obtained, although now the trajectories approach the origin as $t \to \infty$.

These two lines are referred as the *tangents* of the node.

e.g.

$$\begin{aligned} \dot{x} &= 2x + y \\ \dot{y} &= x + 2y \end{aligned}$$

The characteristic equation is

$$s^2 - 4s + 3 = (s - 3)(s - 1) = 0$$

so that the eigenvalues are 1 and 3, and the origin is an unstable node.

The general solution is

$$\begin{aligned} x(t) &= \frac{1}{2}\left((\alpha + \beta)e^{3t} + (\alpha - \beta)e^{t}\right) \\ y(t) &= \frac{1}{2}\left(\alpha + \beta)e^{3t} - (\alpha - \beta)e^{t}\right) \end{aligned}$$

The tangents to the node are given by

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} \quad \text{i.e.} \quad x = y \ ,$$

and

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t} \quad \text{i.e.} \quad x + y = 0 \ .$$

In general,

$$\begin{aligned} x(t) + y(t) &= (\alpha + \beta)e^{3t} = c_1 e^{3t} \\ x(t) - y(t) &= (\alpha - \beta)e^{t} = c_2 e^{t} \\ (x + y) &= c(x - y)^3 \end{aligned}$$

Equal roots.

When $s_1 = s_2$,

$$\begin{aligned} X(s) &= \frac{\alpha(s - a_{22}) + \beta a_{12}}{(s - s_1)^2} \\ &= \frac{\alpha}{s - s_1} + \frac{\alpha(s_1 - a_{22}) + \beta a_{12}}{(s - s_1)^2} \\ Y(s) &= \frac{\beta}{s - s_1} + \frac{\beta(s_1 - a_{11}) + \alpha a_{21}}{(s - s_1)^2} \end{aligned}$$

In the exceptional case in which $a_{11} = a_{22} = s_1$, $a_{12} = a_{21} = 0$, we have $x(t) = \alpha \exp s_1 t$ and $y(t) = \beta \exp s_1 t$, and all the trajectories are straight lines. If $s_1 > 0$, the origin is unstable - *an unstable star node* - while if $s_1 < 0$ we have a *stable star node* at the origin.

However, in general we have

$$\begin{aligned} x(t) &= e^{s_1 t} \left(\alpha + (\alpha(s_1 - a_{22}) + \beta a_{12})t\right) \\ y(t) &= e^{s_1 t} \left(\beta + (\beta(s_1 - a_{11}) + \alpha a_{21})t\right) \end{aligned}$$

If α and β satisfy

$$\begin{aligned} \alpha(s_1 - a_{22}) + \beta a_{12} &= 0 \\ \alpha a_{22} + \beta(s_1 - a_{11}) &= 0 \end{aligned}$$

that is, when $(\alpha, \beta)' = c(a_{12}, a_{22} - s_1)'$ is an eigenvector of the coefficient matrix, we have

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a_{12} \\ a_{22} - s_1 \end{pmatrix} e^{s_1 t}$$

and there is a single tangent to this node.

The pattern of the trajectories has the limiting behaviour of that for a two-tangent node in the limit as the two tangents coallesce.

This behaviour is illustrated in the following example.

$$\begin{aligned}\dot{x} &= x\\ \dot{y} &= x + y\end{aligned}$$

The characteristic polynomial is $(s-1)^2$, but the coefficient matrix is not the unit matrix.

Solving this system, we have

$$\begin{aligned}x &= \alpha e^t\\ \dot{y} &= y + \alpha e^t\\ e^{-t}(\dot{y} - y) &= \alpha\\ e^{-t}y &= \alpha t + \beta\\ y &= \alpha t e^t + \beta e^t\\ &= x \log(x/\alpha) + (\beta/\alpha)x \ (\alpha \neq 0)\end{aligned}$$

The tangent to the node, corresponding to $\alpha = 0$, is the axis $x = 0$. The remaining trajectories are tangent to this line at the origin.

Since the eigenvalue is positive, this node is unstable.

Complex roots.

If the roots are not real, they occur as a conjugate complex pair $s = \sigma \pm i\omega$, and the characteristic polynomial is $(s-\sigma)^2 + \omega^2$.

From the Laplace transform solution we have

$$\begin{aligned}
X(s) &= \frac{\alpha(s-a_{22}) + \beta a_{12}}{(s-\sigma)^2 + \omega^2} \\
&= \frac{\alpha(s-\sigma)}{(s-\sigma)^2 + \omega^2} + \frac{\alpha(\sigma - a_{22}) + \beta a_{12}}{(s-\sigma)^2 + \omega^2} \\
x(t) &= \alpha e^{\sigma t}\cos(\omega t) + \frac{\alpha(\sigma - a_{22}) + \beta a_{12}}{\omega} e^{\sigma t}\sin(\omega t) \\
Y(s) &= \frac{\alpha a_{21} + \beta(s - a_{11})}{(s-\sigma)^2 + \omega^2} \\
y(t) &= \beta e^{\sigma t}\cos(\omega t) + \frac{\alpha a_{21} + \beta(\sigma - a_{11})}{\omega} e^{\sigma t}\sin(\omega t)
\end{aligned}$$

Both x and y oscillate in time, and the solutions circle the origin. If $\sigma > 0$, the oscillations grow in time, and the solution is an *unstable spiral* or *unstable focus*.

If $\sigma < 0$, the oscillations decay with time and the solution is a *stable spiral* or *stable focus*.

If $\sigma = 0$, then both x and y are periodic with period $2\pi/\omega$, and the trajectories are closed curves, about the origin. The equations of these trajectories can be derived directly from the system. Since $\sigma = 0$, $a_{22} = -a_{11}$, and the system is

$$\begin{aligned}
\dot{x} &= a_{11}x + a_{12}y \\
\dot{y} &= a_{21}x - a_{11}y \\
\text{so that} \qquad (a_{21}x - a_{11}y)\dot{x} - (a_{11}x + a_{12}y)\dot{y} &= 0 \\
\frac{1}{2}a_{21}x^2 - a_{11}xy - \frac{1}{2}a_{12}y^2 &= c
\end{aligned}$$

and the trajectories are ellipses centred on the origin.

This structure is called a *centre*. These solutions is stable, since solutions which start near the origin remain near the origin, but not asymptotically stable, since the solutions do not decay to zero.

e.g.

$$\begin{aligned}
\dot{x} &= x - y \\
\dot{y} &= x + y
\end{aligned}$$

The characteristic polynomial is $(s-1)^2 + 1$, and the roots are $s = 1 \pm i$. Since the real part is positive, these solutions form an unstable focus.

The general solution for the trajectories is

$$\begin{aligned}
x(t) &= \alpha e^t \cos t - \beta e^t \sin t \\
&= \sqrt{\alpha^2 + \beta^2} e^t \cos(t + \phi) \\
\text{where} \qquad \tan\phi &= \beta/\alpha \\
y(t) &= \beta e^t \cos t + \alpha e^t \sin t \\
&= \sqrt{\alpha^2 + \beta^2} e^t \sin(t + \phi)
\end{aligned}$$

$$\begin{aligned} \text{In polar co-ordinates,} \quad r(t) &= \sqrt{x^2(t)+y^2(t)} \\ &= \sqrt{\alpha^2+\beta^2}e^t \\ \theta(t) &= \arctan(y(t)/x(t)) \\ &= t+\phi \\ r(t) &= ce^{\theta} \end{aligned}$$

which is the equation of an exponential spiral.

e.g.

$$\begin{aligned} \dot{x} &= x-y \\ \dot{y} &= 5x-y \end{aligned}$$

The characteristic polynomial is $(s-1)(s+1)+5 = s^2+4$, and the roots are $s = \pm 2i$. Since the real part is zero, these solutions form a centre.

The general form for the solutions is

$$\begin{aligned} x(t) &= \alpha \cos 2t + \frac{1}{2}(\alpha-\beta)\sin 2t \\ y(t) &= \beta \cos 2t + \frac{1}{2}(5\alpha-\beta)\sin 2t \end{aligned}$$

which represents the family of ellipses

$$5x^2 - 2xy + y^2 = c\ .$$

The task of sketching these ellipses is simplified when we remember that the tangents are horizontal on the line $y = 5x$, and vertical on the line $y = x$.

Summary.

The nature of the singularity at the origin for the system

$$\begin{aligned} \dot{x} &= a_{11}x + a_{12}y \\ \dot{y} &= a_{21}x + a_{22}y \end{aligned}$$

where $a_{11}a_{22} - a_{12}a_{21} \neq 0$, is determined by the eigenvalues of the coefficient matrix.

If s_1 and s_2 are the roots of

$$s^2 - (a_{11}+a_{22})s + a_{11}a_{22} - a_{12}a_{21} = 0\ ,$$

then

(1) if $s_1 > 0 > s_2$, the origin is an unstable saddle point;
(2) if $s_1 \geq s_2 > 0$, the origin is an unstable node;
(3) if $s_1 \leq s_2 < 0$, the origin is a stable node;
(4) if $s_1 = \sigma + i\omega,\ \sigma > 0,\ \omega \neq 0$, the origin is an unstable focus;
(5) if $s_1 = \sigma + i\omega,\ \sigma < 0,\ \omega \neq 0$, the origin is an stable focus;
(6) if $s_1 = i\omega,\ \omega \neq 0$, the origin is a stable centre.

The origin is asymptotically stable if and only if all the eigenvalues have negative real part.

e.g.

Classify the critical point $(0,0)$, and determine whether it is stable, asymptotically stable, or unstable.

$$\begin{aligned} dx/dt &= 3x - 2y \\ dy/dt &= 2x - 2y \end{aligned}$$

The matrix of the system is

$$A = \begin{pmatrix} 3 & -2 \\ 2 & -2 \end{pmatrix}.$$

The eigenvalues of the problem are given by

$$|\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda - 3 & 2 \\ -2 & \lambda + 2 \end{vmatrix} = (\lambda - 3)(\lambda + 2) + 4 = \lambda^2 - \lambda - 2 = 0$$

$$(\lambda - 2)(\lambda + 1) = 0 \ ; \ \lambda = 2, -1.$$

Since the eigenvalues are of opposite sign, the origin is a saddle point of the system, which is unstable.

e.g.

Classify the critical point $(0,0)$, and determine whether or not it is stable, asymptotically stable, or unstable.

$$\begin{aligned} dx/dt &= \ x - 4y \\ dy/dt &= 4x - 7y \end{aligned}$$

The matrix of the system is

$$A = \begin{pmatrix} 1 & -4 \\ 4 & -7 \end{pmatrix}.$$

The eigenvalues are given by

$$|\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda - 1 & 4 \\ -4 & \lambda + 7 \end{vmatrix} = (\lambda - 1)(\lambda + 7) + 16 = \lambda^2 + 6\lambda + 9 = 0$$

$$(\lambda + 3)^2 = 0 \ ; \ \lambda = -3, -3.$$

Since the eigenvalues are equal and negative, but $A \neq -3I$, the origin is a one-tangent improper node for the system, which is asymptotically stable.

e.g.

Classify the critical point $(0,0)$, and determine whether it is stable, asymptotically stable, or unstable.

$$\begin{aligned} dx/dt &= x - 5y \\ dy/dt &= x - 3y \end{aligned}$$

The matrix of the system is

$$A = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix} .$$

The eigenvalues are given by

$$|\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda - 1 & 5 \\ -1 & \lambda + 3 \end{vmatrix} = (\lambda - 1)(\lambda + 3) + 5 = \lambda^2 + 2\lambda + 2 = 0$$

$$(\lambda + 1)^2 + 1 = 0 \ ; \ \lambda = -1 \pm i.$$

Since the eigenvalues are a complex conjugate pair, the origin is a spiral point of the system, and since the real part (-1) is negative, this is asymptotically stable.

e.g.

Classify the critical point $(0, 0)$, and determine whether it is stable, asymptotically stable, or unstable.

$$\begin{aligned} dx/dt &= 2x - 5y \\ dy/dt &= \ \ x - 2y \end{aligned}$$

The matrix of the system is

$$A = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} .$$

The eigenvalues of the problem are given by

$$|\lambda I - A| = 0$$

$$\begin{vmatrix} \lambda - 2 & 5 \\ -1 & \lambda + 2 \end{vmatrix} = (\lambda - 2)(\lambda + 2) + 5 = \lambda^2 + 1 = 0$$

$$\lambda = \pm i$$

Since the eigenvalues are purely imaginary, the origin is a centre for the system, which is stable but not asymptotically stable.

Almost Linear systems.

If (a, b) is a singular point of the system

$$\dot{x} = P(x, y) \ ; \ \dot{y} = Q(x, y) \ ,$$

we say that the system is *almost linear* in a neighbourhood of (a, b) if the Jacobian

$$P_x(a, b)Q_y(a, b) - Q_x(a, b)P_y(a, b) \neq 0 \ .$$

In this case we can approximate the local phase portrait by that corresponding to the linear system

$$\dot{X} = P_x(a, b)X + P_y(a, b)Y \ ; \ \dot{Y} = Q_x(a, b)X + Q_y(a, b)Y \ .$$

There are three caveats:

Firstly, whereas the phase portrait for the homogeneous system is invariant under dilation, for a non-linear system this only a local approximation, which becomes distorted as we move away from the singularity.

Secondly, the periodic orbits for a centre depend on the exact recurrence of values x and y, and small perturbations may transform a centre into a focus with algebraic growth or decay. The persistence of a centre usually depends on some symmetry in the functions P and Q.

Thirdly, in a similar fashion, a one tangent node in the linear case may become a two tangent node or a spiral in the non-linear case. In this case the stability is unaffected.

e.g.

$$\begin{aligned}\dot{x} &= y \\ \dot{y} &= 1 - x^2 - y^2 \ .\end{aligned}$$

The singular points are $(1, 0)$ and $(-1, 0)$.

Near $(1, 0)$ we have the approximation

$$\begin{gathered}\dot{X} = Y \ ; \ \dot{Y} = -2X \\ 2X\dot{X} + Y\dot{Y} = 0 \ ; \ X^2 + \frac{1}{2}Y^2 = c \ .\end{gathered}$$

These trajectories are ellipses, and the singular point is a centre.

Near $(-1, 0)$ we have the approximation

$$\begin{gathered}\dot{X} = Y \ ; \ \dot{Y} = 2X \\ 2X\dot{X} = Y\dot{Y} \ ; \ X^2 = \frac{1}{2}Y^2 + c \ .\end{gathered}$$

This represents a family of hyperbolas, and the singular point is a saddle point. The separatrices are given (locally) by the asymptotes to the family of hyperbolas, the lines $Y = \pm\sqrt{2}X$.

We can combine these with the previous work to obtain an overall phase portrait for the system.

e.g.

Determine the critical points for each of the following systems.

(a) $$dx/dt = x - xy, \quad dy/dt = y + 2xy$$

$dx/dt = 0$ when $x - xy = 0$, $x(1 - y) = 0$, $x = 0$ or $y = 1$.

$dy/dt = 0$ when $y + 2xy = 0$, $y(1 + 2x) = 0$, $y = 0$ or $x = -1/2$.

Combining these results, the critical points are $(0, 0)$ and $(-1/2, 1)$.

(b) $$dx/dt = x - x^2 - xy, \quad dy/dt = \frac{1}{2}y - \frac{1}{4}y^2 - \frac{3}{4}xy$$

$dx/dt = 0$ when $x = 0$ or $1 - x - y = 0$.
$dy/dt = 0$ when $y = 0$ or $1/2 - y/4 - 3x/4 = 0$.
Considering the possibilities:
When $x = 0$, then either $y = 0$ or $1/2 - y/4 = 0$, $y = 2$. therefore two of the critical points are $(0, 0)$ and $(0, 2)$.
When $x + y = 1$, either $y = 0$, in which case $x = 1$, or $3x + y = 2$, in which case $x = 1/2$ and $y = 1/2$.
Therefore the other two critical points are $(1, 0)$ and $(1/2, 1/2)$.

e.g.

Determine all real critical points of each of the following systems of equations and discuss their type and stability.

(a) $$dx/dt = x + y^2$$
$$dy/dt = x + y$$

At the critical points, $x + y^2 = 0$, so that $x = -y^2$, and $x + y = 0$, $-y^2 + y = 0$, $y = 0$ so that $x = 0$, or $y = 1$ and $x = -1$.

Near $(0, 0)$, the linear part of $x + y^2$ is x, while the linear part of $x + y$ is again $x + y$. Therefore the linear approximation to the system is

$$dx/dt = x$$
$$dy/dt = x + y$$

for which the matrix is
$$A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix},$$
and the eigenvalues $\lambda = 1, 1$, by inspection.

Therefore the linear system has an unstable, one tangent node at the origin, so that the non-linear system will also have an unstable singular point at the origin, but it may be a node or a spiral.

Near $(-1, 1)$, let $x = -1 + X$, $y = 1 + Y$. Then $x + y^2 = -1 + X + 1 + 2Y + Y^2 = X + 2Y + Y^2$, of which the linear part is $X + 2Y$, while $x + y = X + Y$. Therefore the linear approximation to the system is

$$dX/dt = X + 2Y$$
$$dY/dt = X + Y$$

for which the matrix is
$$A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix},$$
and the eigenvalues satisfy $(\lambda - 1)^2 - 2 = 0$, $\lambda = 1 \pm \sqrt{2}$. Since the eigenvalues are real but of opposite sign, $(-1, 1)$ is a saddle point of the linear system and of the non-linear system. A saddle point is unstable.

(b)
$$\begin{aligned} dx/dt &= 1 - xy \\ dy/dt &= x - y^3 \end{aligned}$$

At the critical points, $1-xy=0$, so that $x=1/y$, and also $x-y^3=0$, $y^3=1/y$, $y^4=1$, $y=\pm 1, \pm i$, of which only the first pair are of interest. When $y=1$, $x=1$, and when $y=-1$, $x=-1$.

Near $(1,1)$, set $x=1+X$, and $y=1+Y$. $1-xy=1-(1+X+Y+XY)$ of which the linear part is $-X-Y$. $x-y^3=1+X-(1+3Y+3Y^2+Y^3)$, of which the linear part is $X-3Y$. Therefore the linear approximation to the system is

$$\begin{aligned} dX/dt &= -X - Y \\ dY/dt &= X - 3Y \end{aligned}$$

the matrix of this system is
$$A = \begin{pmatrix} -1 & -1 \\ 1 & -3 \end{pmatrix},$$

and the eigenvalues satisfy

$$(\lambda+1)(\lambda+3)+1 = \lambda^2+4\lambda+4 = 0$$

$\lambda=-2,-2$. Since the eigenvalues are equal and negative, the linear system has an asymptotically stable one-tangent node at $(1,1)$, but the non-linear system may have either an asymptotically stable node or an asymptotically stable spiral at this point.

Near $(-1,-1)$, set $x=-1+X$, and $y=-1+Y$. $1-xy=1-(1-X-Y+XY)$, of which the linear part is $X+Y$. $x-y^3=-1+X-(-1+3Y-3Y^2+Y^3)$, of which the linear part is $X-3Y$. Therefore the linear approximation to the system is

$$\begin{aligned} dX/dt &= X + Y \\ dY/dt &= X - 3Y \end{aligned}$$

the matrix of this system is
$$A = \begin{pmatrix} 1 & 1 \\ 1 & -3 \end{pmatrix},$$

and the eigenvalues satisfy $\lambda^2+2\lambda-4=0$, $\lambda=-1\pm\sqrt{5}$. Since the eigenvalues are real but of opposite sign, both the linear approximation and the non-linear system have an unstable saddle point at $(-1,-1)$.

(c)
$$\begin{aligned} dx/dt &= x - x^2 - xy \\ dy/dt &= 3y - xy - 2y^2 \end{aligned}$$

At the critical points $x-x^2-xy=0$, so that $x=0$ or $x+y=1$, and $3y-xy-2y^2=0$, giving $y=0$ or $x+2y=3$.

Corresponding to $x = 0$, we have the critical points $(0,0)$ and $(0,3/2)$, while corresponding to $x + y = 1$, we have $(1,0)$, and $(-1,2)$.

Near $(0,0)$, $x - x^2 - xy \sim x$, and $3y - xy - 2y^2 \sim 3y$, so that the linear approximation is

$$dx/dt = x \quad ; \quad dy/dt = 3y$$

whose solution is an unstable two-tangent node. Therefore the non-linear system also has an unstable two-tangent node at the origin.

Near $(0,3/2)$, let $x = X$ and $y = 3/2 + Y$.

$$\begin{aligned} x - x^2 - xy &= X - X^2 - X(3/2 + Y) \sim -X/2 \\ 3y - xy - 2y^2 &= 9/2 + 3Y - X(3/2 + Y) - 2(9/4 + 3Y + Y^2) \sim -3X/2 - 3Y \end{aligned}$$

The linear approximation near $(0,3/2)$ is

$$dX/dt = -X/2 \quad : \quad dY/dt = -3X/2 - 3Y$$

for which the eigenvalues are $-1/2$ and -3. $(0,3/2)$ is therefore a stable two-tangent node for both the approximate linear system and the non-linear system.

Near $(1,0)$, let $x = 1 + X$ and $y = Y$.

$$\begin{aligned} x - x^2 - xy &= 1 + X - (1 + 2X + X^2) - Y - XY \sim -X - Y \\ 3y - xy - 2y^2 &= 3Y - Y - XY - 2Y^2 \sim 2Y \end{aligned}$$

The linear approximation near $(1,0)$ is

$$dX/dt = -X - Y \quad ; \quad dY/dt = 2Y$$

for which the eigenvalues are -1 and 2. Therefore both the linear approximation and the non-linear system have a saddle point at $(1,0)$.

Near $(-1,2)$, let $x = -1 + X$ and $y = 2 + Y$.

$$\begin{aligned} x - x^2 - xy &= -1 + X - (1 - 2X + X^2) - (-2 + 2X - Y + XY) \sim X + Y \\ 3y - xy - 2y^2 &= 6 + 3Y - (-2 + 2X - Y + XY) - 2(4 + 4Y + Y^2) \sim -2X - 4Y \end{aligned}$$

The matrix of the linear approximation near $(-1,2)$ is

$$A = \begin{pmatrix} 1 & 1 \\ -2 & -4 \end{pmatrix},$$

whose eigenvalues satisfy $(\lambda - 1)(\lambda + 4) + 2 = 0$, $\lambda^2 + 3\lambda - 2 = 0$, $\lambda = (-3 \pm \sqrt{17})/2$. Since the eigenvalues are real but of opposite signs, the linear approximation and the non-linear system have a saddle point at $(-1,2)$.

(d)
$$\begin{aligned} dx/dt &= 1 - y \\ dy/dt &= x^2 - y^2 \end{aligned}$$

The critical points correspond to $1 - y = 0$, $y = 1$, and $x^2 - y^2 = 0$, $x^2 = 1$, $x = \pm 1$.

Near $(1, 1)$, let $x = 1 + X$ and $y = 1 + Y$.

$$1 - y = -Y \text{ and } x^2 - y^2 = 1 + 2X + X^2 - (1 + 2Y + Y^2) \sim 2X - 2Y$$

The matrix of the linear approximation is

$$A = \begin{pmatrix} 0 & -1 \\ 2 & -2 \end{pmatrix},$$

for which the eigenvalues satisfy $\lambda(\lambda + 2) + 2 = 0$, $\lambda = -1 \pm i$. Therefore the linear approximation and the non-linear system have a stable spiral at $(1, 1)$.

Near $(-1, 1)$, let $x = -1 + X$ and $y = 1 + Y$.

$$1 - y = -Y \text{ and } x^2 - y^2 = 1 - 2X + X^2 - (1 + 2Y + Y^2) \sim -2X - 2Y$$

The matrix of the linear approximation is

$$A = \begin{pmatrix} 0 & -1 \\ -2 & -2 \end{pmatrix},$$

for which the eigenvalues satisfy $\lambda(\lambda + 2) - 2 = 0$, $\lambda = -1 \pm \sqrt{3}$. Therefore the linear approximation and the non-linear system have a saddle point at $(-1, 1)$.

Isolated periodic solutions.

As well as families of periodic solutions associated with centres, second order non-linear systems may have *limit cycles*, which are isolated periodic solutions.

Consider, for example, Van der Pol's equation

$$\ddot{x} + (x^2 - 1)\dot{x} + x = 0 \ ,$$

or, in systems form

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= \ -x - (x^2 - 1)y \end{aligned}$$

The only singular point of the system is $(x, y) = (0, 0)$, near which we have the linear approximation

$$\dot{X} = Y \ ; \ \dot{Y} = -X + Y \ .$$

The characteristic equation is $s^2 - s + 1 = 0$, so that the eigenvalues are $s = 1/2 \pm i\sqrt{3}/2$, and the origin is an unstable focus.

On the other hand, by considering the isocline $\dot{y} = 0$, i.e. $y = x/(1 - x^2)$, we see that solutions starting far from the origin spiral inwards. Since trajectories cannot cross, there must be some boundary between these two regions. This boundary is a closed trajectory, representing a periodic solution, called a limit cycle.

Limit cycles of this type are important (in, for example, radio transmission) because they represent physically realisable periodic solutions of fixed amplitude and frequency.

Liapunov functions.

The process of linearisation about a singular point only provides information about the stability of the singular point if it is almost linear.

To determine the stability of a general singular point, an alternative procedure is adopted.

A singular point is asymptotically stable if all trajectories in some neighbourhood of the point approach the point as $t \to \infty$. Ideally this would mean that $X^2 + Y^2$ decreases monotonically to zero along the trajectories. Unfortunately, consideration of skewed spirals shows that this may not be the case. The idea behind Liapunov functions is to replace the Euclidean distance $X^2 + Y^2$ with an appropriate generalisation.

Definition. A Liapunov function for the point (a, b), is a function $V(x, y)$ with continuous partial derivatives such that

(i) $V(a, b) = 0$;

(ii) there is a deleted neighbourhood of (a, b) throughout which $V(x, y) > 0$.

This means that for some $0 < c < C$, the level curves $V(x, y) = c$ form a set of nested contours about (a, b). Our criterion for stability is now that $V(x, y)$ should decrease monotonically along trajectories in a neighbourhood of (a, b).

If $x(t), y(t)$ is a trajectory of the system, then along this trajectory the Liapunov function is $V(x(t), y(t))$, which is monotonic decreasing if $dV/dt \le 0$.

By the chain rule,

$$\frac{dV}{dt} = \frac{\partial V}{\partial x}\frac{dx}{dt} + \frac{\partial V}{\partial y}\frac{dy}{dt} = V_x P(x, y) + V_y Q(x, y) \ .$$

Hence we can determine whether V is or is not monotonically decreasing without solving the system of differential equations (or any approximation to it).

(For simplicity, we will assume that $(a, b) = (0, 0)$.)

e.g.

Consider the equations for the damped oscillations of a pendulum

$$\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= \ -\frac{g}{l}\sin\theta - \epsilon\omega \ ,\end{aligned}$$

and the function $V(\theta, \omega) = mgl(1 - \cos\theta) + \frac{1}{2}ml^2\omega^2$, which represents the energy of the system.

$V(0, 0) = 0$, and for $-2\pi < \theta < 2\pi$, and $\omega \in \mathbb{R}$, $V(\theta, \omega) > 0$ otherwise. Therefore V has the required properties to be a Liapunov function in a neighbourhood of $(0, 0)$.

For small values of θ, $\cos\theta \simeq 1 - \frac{1}{2}\theta^2$, so that $V(\theta, \omega) \simeq \frac{1}{2}ml(g\theta^2 + l\omega^2)$, and the level contours of V are approximately ellipses near $(0, 0)$.

If $\theta(t), \omega(t)$ is a trajectory of the system near $(0, 0)$, then

$$\begin{aligned}\frac{dV(\theta(t), \omega(t))}{dt} &= \frac{\partial V}{\partial \theta}\frac{d\theta}{dt} + \frac{\partial V}{\partial \omega}\frac{d\omega}{dt} \\ &= mgl\sin(\theta)\omega - ml^2\omega\left(\frac{g}{l}\sin(\theta) + \epsilon\omega\right) \\ &= \ -\epsilon ml^2\omega^2 \le 0\end{aligned}$$

Since $dV/dt \le 0$ along trajectories, these trajectories run *downhill* to the stationary point $(0, 0)$, which is therefore asymptotically stable.

e.g.

Consider the system

$$\dot{x} = y^3 \ ; \ \dot{y} = -x^3$$

which has a singular point at $(0,0)$ which is **not** almost linear.

If we choose $V(x,y) = x^4 + y^4$, this is obviously suitable as a Liapunov function about $(0,0)$.

If $x(t), y(t)$ is a trajectory for the system, then along this trajectory

$$\frac{dV}{dt} = 4x^3\dot{x} + 4y^3\dot{y} = 4x^3y^3 - 4y^3x^3 = 0 \ .$$

In this case the trajectory remains on the (closed) contour $V = c$, and so the solutions of this system are all periodic. The origin is stable but not asymptotically stable.

e.g.

Consider the system

$$\dot{x} = y + x^3 \ ; \ \dot{y} = -x + y^3 \ .$$

This system has a singular point at $(0,0)$, for which the almost linear system has a centre.

However, if we choose $V(x,y) = x^2 + y^2$, we find that along trajectories

$$\frac{dV}{dt} = 2x\dot{x} + 2y\dot{y} = 2xy + 2x^4 - 2yx + 2y^4 > 0 \ \text{ for } \ (x,y) \neq (0,0) \ .$$

Since the trajectories are moving further away from the origin with time, the origin is in fact an unstable spiral point.

From these examples it is apparent that the major difficulty with this method is in deriving a suitable Liapunov function.

For systems in which the functions P and Q are polynomials, it is usual to try to find V in the form

$$V(x,y) = ax^2 + bxy + cy^2$$

where a, b, c are chosen so that $dV/dt \leq 0$ along trajectories.

If such a form is found, we need to determine its behaviour. Completing the square, we find

$$\begin{aligned} V(x,y) &= a\left(x^2 + \frac{b}{a}xy + \left(\frac{b}{2}ay\right)^2\right) + \left(c - \frac{b^2}{4a}\right)y^2 \\ &= a\left(x + \frac{b}{2a}y\right)^2 + \frac{4ac - b^2}{4a}y^2 \end{aligned}$$

If $a > 0$ and $4ac > b^2$, $V(x,y) > 0$ for all $(x,y) \neq (0,0)$. The function V is positive definite.

If $a < 0$ and $4ac > b^2$, $V(x,y) < 0$ for all $(x,y) \neq (0,0)$. The function V is negative definite.

If we obtain a difference of squares, the form is indefinite.

By construction, trajectories of the system run downhill, so that the origin is stable if the form V is positive definite, an unstable node or spiral if V is negative definite, and a saddle point if V is indefinite.

e.g.

Consider the system

$$\dot{x} = y^3 \ ; \ \dot{y} = x^3 \ .$$

If we assume $V(x,y) = ax^2 + bxy + cy^2$, then along trajectories

$$\begin{aligned}\frac{dV}{dt} &= (2ax + by)\dot{x} + (bx + 2cy)\dot{y} \\ &= 2axy^3 + by^4 + bx^4 + 2cx^3y \\ &= -(x^4 + y^4) < 0 \quad \text{if} \quad V = -xy \ . \\ -xy &= \left(\frac{x-y}{2}\right)^2 - \left(\frac{x+y}{2}\right)^2\end{aligned}$$

so that V is an indefinite form, and the origin is an unstable saddle point.

EXAMPLES OF EXAM QUESTIONS

1. By constructing a suitable Liapunov function of the form $ax^2 + cy^2$, where a and c are to be determined, show that the origin is an asymptotically stable critical point of the system

$$\begin{aligned}\frac{dx}{dt} &= -x^3 + xy^2 \\ \frac{dy}{dt} &= -2x^2y - y^3 \ .\end{aligned}$$

2. Use the Laplace Transform to find the general solution of the system

$$\begin{aligned}\frac{dx}{dt} &= 3x - 2y \\ \frac{dy}{dt} &= 2x - 2y \ .\end{aligned}$$

Sketch the solution curves in the neighbourhood of the origin.
What type of singular point does this system have at the origin?

3. Consider the system

$$\begin{aligned}\dot{x} &= x \\ \dot{y} &= x^2 + y^2 - 1\end{aligned}$$

Determine the singular points of the system, and the approximate form of the trajectories in the neighbourhoods of these points.
Hence sketch the phase portrait for the system.

4. Determine the values of α for which the singular point $(0,0)$ of the system

$$\begin{aligned}\dot{x} &= -3x + \alpha y \\ \dot{y} &= 2x + y\end{aligned}$$

is stable.