Systems of differential equations

Consider to begin with the 'first order system'; i.e. the single differential equation

$$\dot{x} = f(t, x) \ .$$

If we specify t_0 and a, then $\phi(t)$ is a solution of the *initial value problem* provided

(i)
$$\frac{d\phi}{dt} \equiv f(t, \phi(t))$$

on some interval containing t_0, and

(ii)
$$\phi(t_0) = a \ .$$

Usually the domain of definition of the solution depends on a. For example

$$\dot{x}(t) = x^2 \quad ; \quad x(0) = a$$

has the solution

$$x(t) = \frac{a}{1 - at}$$

which is defined on $(-\infty, 1/a)$ if $a > 0$, and on $(1/a, \infty)$ if $a < 0$.

Similarly, for a second order system

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, x_2) \\ \dot{x}_2 &= f_2(t, x_1, x_2) \end{aligned}$$

if we specify t_0, a_1, a_2, then $(\phi_1(t), \phi_2(t))'$ is a solution pair for the initial value problem if

(i)
$$\begin{aligned} \frac{d\phi_1}{dt} &\equiv f_1(t, \phi_1(t), \phi_2(t)) \\ \frac{d\phi_2}{dt} &\equiv f_2(t, \phi_1(t), \phi_2(t)) \end{aligned}$$

on some interval containing t_0, and

(ii)
$$\phi_1(t_0) = a_1 \qquad \phi_2(t_0) = a_2 \ .$$

In general, an n^{th} order system of differential equations is

$$\begin{aligned} \dot{x}_1 &= f_1(t, x_1, x_2, \ldots, x_n) \\ \dot{x}_2 &= f_2(t, x_1, x_2, \ldots, x_n) \\ &\ldots \\ \dot{x}_n &= f_n(t, x_1, x_2, \ldots, x_n) \end{aligned}$$

or, in concise notation

$$\underline{\dot{x}} = \underline{f}(t, \underline{x}) \ .$$

If we specify $t_0, a_1, a_2, \ldots, a_n \qquad (t_0, \underline{a})$ we have an n^{th} order initial value problem, and $\underline{\phi}(t) = (\phi_1(t), \phi_2(t), \ldots, \phi_n(t))'$ is a solution provided

(i) $$\frac{d\phi}{dt} \equiv \underline{f}(t, \underline{\phi}(t))$$

on some interval containing t_0 and

(ii) $$\underline{\phi}(t_0) = \underline{a}\ .$$

For example, consider the third order system

$$\begin{aligned} \dot{x}_1 &= x_2^2 - 1 \\ \dot{x}_2 &= x_2 x_3 - x_1 \\ \dot{x}_3 &= x_3^2 - 2x_2 - 2 \end{aligned}$$

$$\underline{X}(t) = (2, 1, 2)'$$

is a solution satisfying $\underline{x}(0) = (2, 1, 2)'$,

$$\underline{Y}(t) = (8t/(1 + 4t^2), (3 - 4t^2)/(1 + 4t^2), -8t/(1 + 4t^2))'$$

is a solution satisfying $\underline{x}(0) = (0, 3, 0)'$, and

$$\underline{Z}(t) = (1 - \tanh t, - \tanh t, 1 - \tanh t)'$$

is a solution satisfying $\underline{x}(0) = (1, 0, 1)'$.

Note that all these solutions are defined for $-\infty < t < \infty$.

Existence Theorem. (Cauchy)

If the functions $f_1(t, \underline{x}), \ldots, f_n(t, \underline{x})$ are all continuous in all the variables in a neighbourhood of $t_0, \underline{a}$, then the initial value problem $\underline{\dot{x}} = \underline{f}(t, \underline{x})$, $\underline{x}(t_0) = \underline{a}$, has a solution.

For example, in the third order system above, all the functions are polynomials in the variables, and hence are continuous for all their values. Hence this system has a solution for every choice of t_0, a_1, a_2, a_3.

Uniqueness Theorem. (Picard)

If, in addition, the n^2 partial derivatives $\partial f_i / \partial x_j$ are also continuous in a neighbourhood of $t_0, \underline{a}$, then the solution is (locally) unique.

For the third order system above, the partial derivatives are

$$\begin{matrix} 0 & 2x_2 & 0 \\ -1 & x_3 & x_2 \\ 0 & -2 & 2x_3 \end{matrix}$$

all of which are everywhere continuous. Hence the solutions of the initial value problems for this system are unique.

Given an n^{th} order system, let us fix t_0. For each $\underline{a}$ we can calculate numerically the appropriate solution $\underline{\phi} \equiv \underline{\phi}(t, a_1, a_2, \ldots, a_n)$.

[For example: Choose h small and positive, and denote $t_n = t_0 + nh$.

The approximate value $\underline{x}_n$ of $\underline{x}(t_n)$ can be calculated recursively for $n = 0, 1, \ldots, N$ by

$$
\begin{aligned}
\underline{x}_0 &= \underline{a} \\
\underline{\xi}_{n+1} &= \underline{x}_n + h\underline{f}(t_n, \underline{x}_n) \\
\underline{x}_{n+1} &= \frac{1}{2}\left(\underline{x}_n + \underline{\xi}_{n+1} + h\underline{f}(t_{n+1}, \underline{\xi}_{n+1})\right) \ .
\end{aligned}
$$

This numerical procedure is called a *predictor-corrector* method; specifically, the 'Method of Heun'.]

In theory, an infinite computer with symbolic manipulation could obtain a general formula $\underline{\phi}(t, \underline{a})$, into which we could substitute particular values of $\underline{a}$ to obtain particular solutions. More generally, if we can find a function $\underline{\phi}(t, \underline{\alpha})$, involving n parameters $\underline{\alpha}$ and such that the solutions of particular initial value problems can be found by determining particular values of these parameters, then $\underline{\phi}$ is called a 'general solution' of the differential system.

For example, if we substitute

$$
\begin{aligned}
x_1 &= a_1 + a_{11}t + a_{12}t^2 + \ldots \\
x_2 &= a_2 + a_{21}t + a_{22}t^2 + \ldots \\
x_3 &= a_3 + a_{31}t + a_{32}t^2 + \ldots
\end{aligned}
$$

into the third order system above and equate the coefficients, we find

$$
\begin{array}{ccc}
a_{11} = a_2^2 - 1 & a_{21} = a_2a_3 - a_1 & a_{31} = a_3^2 - 2a_2 - 2 \\
a_{12} = a_2a_{21} = a_2^2a_3 - a_1a_2 & a_{22} = \frac{1}{2}(a_2a_{31} + a_3a_{21} - a_{11}) & a_{32} = a_3a_{31} - a_{21}
\end{array}
$$

so that, formally at least, a general solution is

$$
\begin{aligned}
x_1 &= a_1 + (a_2^2 - 1)t + (a_2^2a_3 - a_1a_2)t^2 + \ldots \\
x_2 &= a_2 + (a_2a_3 - a_1)t + \frac{1}{2}(2a_2a_3^2 - 3a_2^2 - 2a_2 - a_1a_3 + 1)t^2 + \ldots \\
x_3 &= a_3 + (a_3^2 - 2a_2 - 2)t + (a_3^3 - 3a_2a_3 - 2a_3 + a_1)t^2 + \ldots
\end{aligned}
$$

Linear systems.

An n^{th} order system is linear if it can be written in the form

$$
\begin{aligned}
\dot{x}_1 &= a_{11}(t)x_1(t) + a_{12}(t)x_2(t) + \cdots + a_{1n}(t)x_n(t) + b_1(t) \\
\dot{x}_2 &= a_{21}(t)x_1(t) + a_{22}(t)x_2(t) + \cdots + a_{2n}(t)x_n(t) + b_2(t) \\
&\ldots \\
\dot{x}_n &= a_{n1}(t)x_1(t) + a_{n2}(t)x_2(t) + \cdots + a_{nn}(t)x_n(t) + b_n(t)
\end{aligned}
$$

or, using matrix notation,

$$\underline{\dot{x}} = A(t)\underline{x}(t) + \underline{b}(t) \ .$$

If the vector $\underline{b}$ is identically $\underline{0}$, we say that the system is **homogeneous**. Otherwise, we have an **inhomogeneous** system. The homogeneous system $\dot{\underline{x}} = A\underline{x}$ which corresponds to the inhomogeneous system $\dot{\underline{x}} = A\underline{x} + \underline{b}$ is referred to as the **reduced** system.

Since the variables x_j appear in the right-hand side as linear functions, the right-hand side is always continuous in the dependent variables. Hence, the existence condition reduces to the requirement that the functions $a_{ij}(t)$ and $b_i(t)$ be continuous in some neighbourhood of t_0. Furthermore, the partial derivatives $\partial f_i/\partial x_j$ are the coefficients $a_{ij}(t)$, so that the existence condition also guarantees uniqueness.

We will assume from now on that we are operating in some interval $I = (\alpha, \beta)$, where α may be $-\infty$ and β may be ∞, such that all the coefficient functions are continuous functions of t on I, so that the existence and uniqueness theorems apply.

The possibly more familiar n^{th} order linear differential equation

$$\frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}n}{dt^{n-1}} + \cdots + p_0(t)x = f(t)$$

can be written as a system by setting

$$x_1 = x\ ,\ x_2 = \dot{x}\ ,\ \ldots\ ,\ x_n = \frac{d^{n-1}x}{dt^{n-1}}$$

to obtain

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \\ \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ . & . & . & \ldots & . \\ 0 & 0 & 0 & \ldots & 1 \\ -p_0 & -p_1 & -p_2 & \ldots & -p_{n-1} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ \\ x_{n-1} \\ x_n \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \\ 0 \\ f \end{pmatrix}$$

The converse process is not possible in general.

Linear systems are important because of the structure of their solution sets.

Consider firstly the homogeneous equation $\dot{\underline{x}} = A\underline{x}$.

If $\underline{X}$ and $\underline{Y}$ are solutions of this equation on the same interval I, then

$$c_1\dot{\underline{X}} + c_2\dot{\underline{Y}} = c_1 A\underline{X} + +c_2 A\underline{Y} = A(c_1\underline{X} + c_2\underline{Y})\ .$$

That is, any linear combination of solutions is also a solution.

Suppose that we have found n different solutions $\underline{Y}_1, \ldots, \underline{Y}_n$. Then

$$\underline{X} = c_1\underline{Y}_1 + \cdots + c_n\underline{Y}_n$$

is also a solution, and this solution form involves n arbitrary parameters $\underline{c}$. Therefore, provided we can choose these parameters to satisfy **any** initial conditions, we have constructed a general solution to the system.

Denote $\underline{Y}_i = (Y_{1i}, \ldots, Y_{ni})'$, and choose $t_0 \in I$.

$\underline{X} = c_1\underline{Y}_1 + \cdots + c_n\underline{Y}_n$ will be a general solution provided, given an arbitrary set $\underline{a}$ of initial values, we can choose $\underline{c} = (c_1, \ldots, c_n)'$ so that $\underline{X}(t_0) = \underline{a}$. Expanding these equations we have

$$\begin{pmatrix} Y_{11}(t_0) & Y_{12}(t_0) & \ldots & Y_{1n}(t_0) \\ Y_{21}(t_0) & Y_{22}(t_0) & \ldots & Y_{2n}(t_0) \\ . & . & \ldots & . \\ Y_{n1}(t_0) & Y_{n2}(t_0) & \ldots & Y_{nn}(t_0) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ . \\ c_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ . \\ a_n \end{pmatrix}\ .$$

The condition that this set of equations be soluble for every possible choice of $\underline{a}$ is that the determinant

$$\begin{vmatrix} Y_{11}(t_0) & Y_{12}(t_0) & \dots & Y_{1n}(t_0) \\ Y_{21}(t_0) & Y_{22}(t_0) & \dots & Y_{2n}(t_0) \\ . & . & \dots & . \\ Y_{n1}(t_0) & Y_{n2}(t_0) & \dots & Y_{nn}(t_0) \end{vmatrix}$$

should be non-zero. This determinant is called the **Wronskian** of the solutions $\underline{Y}_1, \dots, \underline{Y}_n$ at the point t_0, and is denoted by $W[\underline{Y}_1, \dots, \underline{Y}_n](t_0)$. When the Wronskian is non-zero at t_0 we say that the solutions are independent at t_0, while if the Wronskian vanishes they are dependent.

If the Wronskian of the solutions vanishes at t_0, then there are non-trivial constants $\underline{c}$ such that $c_1\underline{Y}_1(t_0) + \cdots + c_n\underline{Y}_n(t_0) = \underline{0}$. But this means that $\underline{X} = c_1\underline{Y}_1 + \cdots + c_n\underline{Y}_n$ is **the** solution of the initial value problem $\dot{\underline{X}} = A\underline{X}$, $\underline{X}(t_0) = \underline{0}$; i.e. $\underline{X}(t) \equiv \underline{0}$ on I. Hence,

$$\begin{pmatrix} Y_{11}(t_1) & Y_{12}(t_1) & \dots & Y_{1n}(t_1) \\ Y_{21}(t_1) & Y_{22}(t_1) & \dots & Y_{2n}(t_1) \\ . & . & \dots & . \\ Y_{n1}(t_1) & Y_{n2}(t_1) & \dots & Y_{nn}(t_1) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ . \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ . \\ 0 \end{pmatrix}$$

for every $t_1 \in I$, and the Wronskian vanishes identically on the interval. This means that a set of solutions is either dependent or independent on the **entire** interval I.

This is a property of solution sets for linear homogeneous equations. It does not hold, for example, for the three solutions of the non-linear system considered above. It should also be noted that that, while the knowledge of three independent solutions of a third order linear system is sufficient to determine the general solution, the same is not the case for the non-linear system above. We have three independent solutions, but this is of no use in determining a general solution.

If we are dealing with an n^{th} order linear equation

$$\frac{d^n x}{dt^n} + p_{n-1}(t)\frac{d^{n-1}x}{dt^{n-1}} + \cdots + p_0(t)x = 0 \ ,$$

then for any solution $\phi(t)$ of this equation the corresponding linear system has the solution $\underline{\Phi}(t) = (\phi(t), \dot{\phi}(t), \dots, \phi^{(n-1)}(t))'$, so that, if $\phi_1(t), \dots, \phi_n(t)$ are n solutions of the equation, their Wronskian is

$$\begin{vmatrix} \phi_1(t) & \phi_2(t) & \dots & \phi_n(t) \\ \dot{\phi}_1(t) & \dot{\phi}_2(t) & \dots & \dot{\phi}_n(t) \\ . & . & \dots & . \\ \phi_1^{(n-1)}(t) & \phi_2^{(n-1)}(t) & \dots & \phi_n^{(n-1)}(t) \end{vmatrix} .$$

This form is known as the Wronskian of the n functions $\phi_1, \dots, \phi_n$, denoted by $W(\phi_1, \dots, \phi_n)$.

Consider the following two exercises from Boyce and DiPrima.

p129 - 4-6. Compute the Wronskians of the following pairs of functions:
4. x, e^x 5. $e^x \sin x, e^x \cos x$ 6. $\cos^2 x, 1 + \cos 2x$

(4) $$W(x, xe^x) = \begin{vmatrix} x & xe^x \\ 1 & (x+1)e^x \end{vmatrix} = x^2 e^x.$$

(5) $$W(e^x \sin x, e^x \cos x) = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x(\sin x + \cos x) & e^x(\cos x - \sin x) \end{vmatrix}$$
$$= e^{2x}(\sin x \cos x - \sin^2 x - \cos x \sin x - \cos^2 x) = -e^{2x}.$$

(6) $$W\left(\cos^2 x, 1 + \cos 2x\right) = W(\cos^2 x, 2\cos^2 x) = 0$$

p126 - 4. Prove that the functions y_1 and y_2 are linearly independent solutions of the given differential equation.

$$x^2 y'' + xy' - 4y = 0, \qquad x > 0; \qquad y_1(x) = x^2, \quad y_2(x) = x^{-2}$$

$$y_1' = 2x,\ y_1'' = 2,\ x^2 y_1'' + x y_1' - 4y_1 = 2x^2 + 2x^2 - 4x^2 = 0$$
$$y_2' = -2x^{-3},\ y_2'' = 6x^{-4},\ x^2 y_2'' + x y_2' - 4y_2 = 6x^{-2} - 2x^{-2} - 4x^{-2} = 0$$

Therefore both y_1 and y_2 are solutions of the equation for $x > 0$.

$$W(x^2, x^{-2}) = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = -2x^{-1} - 2x^{-1} = -4x^{-1} \neq 0 \text{ for } \ x > 0.$$

Since the Wronskian is non-zero, the solutions are independent.

Abel's Identity.

Consider for simplicity the second order system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} .$$

If $\underline{\phi}(t) = (\phi_1(t), \phi_2(t))'$ and $\underline{\psi}(t) = (\psi_1(t), \psi_2(t))'$ are two solutions of this system on I, then we have

$$W[\underline{\phi}, \underline{\psi}] = \phi_1\psi_2 - \phi_2\psi_1$$
$$\frac{dW}{dt} = \dot{\phi}_1\psi_2 + \phi_1\dot{\psi}_2 - \dot{\phi}_2\psi_1 - \phi_2\dot{\psi}_1$$
$$= (a_{11}\phi_1 + a_{12}\phi_2)\psi_2 + \phi_1(a_{21}\psi_1 + a_{22}\psi_2) - (a_{21}\phi_1 + a_{22}\phi_2)\psi_1 - \phi_2(a_{11}\psi_1 + a_{12}\psi_2)$$
$$= (a_{11} + a_{22})(\phi_1\psi_2 - \phi_2\psi_1) = (a_{11} + a_{22})W \ . \qquad \blacksquare$$

Hence, $W[\underline{\phi}, \underline{\psi}](t) = W[\underline{\phi}, \underline{\psi}](t_0) \exp(\int_{t_0}^t (a_{11}(s) + a_{22}(s))\, ds)$. Since the functions a_{11} and a_{22} are continuous on I, the integral is finite for $t \in I$, and hence the exponential never vanishes on I. Abel's formula therefore provides an alternate proof that the solution sets are either dependent everywhere on I or independent everywhere on I. It also shows that the functional form of the Wronskian of the solutions is determined to within a constant multiple by the system itself, irrespective of the particular solution set.

For an n^{th} order system, the same procedure can be used to show that

$$W[\underline{Y}_1, \ldots, \underline{Y}_n](t) = W[\underline{Y}_1, \ldots, \underline{Y}_n](t_0) \exp\left(\int_{t_0}^{t} (a_{11}(s) + \cdots + a_{nn}(s))\, ds \right) .$$

When the system has been derived from an n^{th} order equation, most of the diagonal elements are zero, and the formula reduces to

$$W[\phi_1, \ldots, \phi_n](t) = W[\phi_1, \ldots, \phi_n](t_0) \exp\left(- \int_{t_0}^{t} p_{n-1}(s)\, ds \right) .$$

The general solution of the system on the interval I can be represented in the concise form $\underline{x} = \mathcal{X}(t)\underline{c}$, where $\mathcal{X}(t)$ is the $n \times n$ matrix whose columns are the vectors $\underline{Y}_i$. $\mathcal{X}$ is referred to as a **fundamental matrix** for the system. $\mathcal{X}$ is a solution of the matrix differential equation $\dot{X} = AX$. Different choices of solution sets lead to different forms of fundamental matrix. However, if we specify that we require $X(t_0) = I$, we obtain the unique form $\Phi(t, t_0) = \mathcal{X}(t)\mathcal{X}^{-1}(t_0)$ irrespective of the choice of $\mathcal{X}$. The solution of $\dot{\underline{x}} = A\underline{x}$ such that $\underline{x}(t_0) = \underline{a}$ is $\underline{x} = \Phi(t, t_0)\underline{a}$.

Example.

Consider the third order differential equation

$$\dddot{x} + 4\dot{x} = 0 .$$

We can convert this to a linear system by setting $x_1 = x$, $x_2 = \dot{x}$ and $x_3 = \ddot{x}$. Then

$$\dot{x}_1 = \dot{x} = x_2 , \quad \dot{x}_2 = \ddot{x} = x_3 , \quad \dot{x}_3 = \dddot{x} = -4\dot{x} = -4x_2 ,$$

or, in matrix form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -4 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{aligned} \underline{Y}_1(t) &= (\cos^2(t), -2\sin(t)\cos(t), 2\sin^2(t) - 2\cos^2(t))' \\ \underline{Y}_2(t) &= (\sin^2(t), 2\sin(t)\cos(t), 2\cos^2(t) - 2\sin^2(t))' \\ \underline{Y}_3(t) &= (\cos(2t), -2\sin(2t), -4\cos(2t))' \end{aligned}$$

are solutions of the system. The Wronskian of these solutions is

$$\begin{aligned} &\begin{vmatrix} \cos^2(t) & \sin^2(t) & \cos(2t) \\ -2\sin(t)\cos(t) & 2\sin(t)\cos(t) & -2\sin(2t) \\ 2\sin^2(t) - 2\cos^2(t) & 2\cos^2(t) - 2\sin^2(t) & -4\cos(2t) \end{vmatrix} \\ &= \begin{vmatrix} 1 & \sin^2(t) & \cos(2t) \\ 0 & \sin(2t) & -2\sin(2t) \\ 0 & 2\cos(2t) & -4\cos(2t) \end{vmatrix} = 0 \end{aligned}$$

so that these solutions are linearly dependent.

A fourth solution is

$$\underline{Y}_4(t) = (\sin(2t), 2\cos(2t), -4\sin(2t)\ .$$

The Wronskian of $\underline{Y}_1, \underline{Y}_2, \underline{Y}_4$ is

$$\begin{vmatrix} \cos^2(t) & \sin^2(t) & \sin(2t) \\ -2\sin(t)\cos(t) & 2\sin(t)\cos(t) & 2\cos(2t) \\ 2\sin^2(t) - 2\cos^2(t) & 2\cos^2(t) - 2\sin^2(t) & -4\sin(2t) \end{vmatrix}$$
$$= \begin{vmatrix} 1 & \sin^2(t) & \sin(2t) \\ 0 & \sin(2t) & 2\cos(2t) \\ 0 & 2\cos(2t) & -4\sin(2t) \end{vmatrix} = -4$$

so that this set of solutions is independent, and a general solution of the system can be written

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \cos^2(t) & \sin^2(t) & \sin(2t) \\ -2\sin(t)\cos(t) & 2\sin(t)\cos(t) & 2\cos(2t) \\ 2\sin^2(t) - 2\cos^2(t) & 2\cos^{(}t) - 2\sin^2(t) & -4\sin(2t) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

In particular, $x_1 = c_1\cos^2(t) + c_2\sin^2(t) + c_3\sin(2t)$, so that a general solution of the original third order differential equation is $x = c_1\cos^2(t) + c_2\sin^2(t) + c_3\sin(2t)$.

When $t = 0$, the fundamental matrix has the value

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 2 \\ -2 & 2 & 0 \end{pmatrix}$$

whose inverse is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}\ .$$

The fundamental matrix which reduces to the unit matrix when $t = 0$ is

$$\begin{pmatrix} \cos^2(t) & \sin^2(t) & \sin(2t) \\ -2\sin(t)\cos(t) & 2\sin(t)\cos(t) & 2\cos(2t) \\ 2\sin^2(t) - 2\cos^2(t) & 2\cos^2(t) - 2\sin^2(t) & -4\sin(2t) \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \end{pmatrix}$$
$$= \begin{pmatrix} 1 & \frac{1}{2}\sin(2t) & \frac{1}{2}\sin^2(t) \\ 0 & \cos(2t) & \sin(t)\cos(t) \\ 0 & -2\sin(2t) & \cos(2t) \end{pmatrix}\ .$$

Example 2.

The second order system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} (t+1) & (1 - t - t^2) \\ 1 & -t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

has the solutions

$$\underline{Y}_1 = (t, 1)'$$
$$\underline{Y}_2 = ((t+1)e^t, e^t)'$$

Their Wronskian is

$$\begin{vmatrix} t & (t+1)e^t \\ 1 & e^t \end{vmatrix} = te^t - (t+1)e^t = -e^t$$

Note that $a_{11}(s) + a_{22}(s) = (s+1) - s = 1$, $\exp(\int_0^t 1\, ds) = e^t$. Since the Wronskian is never zero, these solutions are independent.

When $t = 0$, the fundamental matrix has the value

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{whose inverse is} \quad \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} .$$

The fundamental matrix which reduces to I when $t = 0$ is

$$\begin{pmatrix} t & (t+1)e^t \\ 1 & e^t \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} (t+1)e^t - t & t \\ e^t - 1 & 1 \end{pmatrix} .$$

The method of successive approximations.

Consider the general problem of constructing the solution $\Phi(t, t_0)$ for the system $\dot{\underline{x}} = A(t)\underline{x}$. One theoretical approach is the *method of successive approximations*, which was introduced by Picard.

If we set $\Phi_0 = I$, the unit matrix, and define successively

$$\Phi_n(t) = I + \int_{t_0}^{t} A(s)\Phi_{n-1}(s)\, ds \ ,$$

then the successive approximations Φ_n converge to the required solution.

For example, if we consider the first order system

$$\dot{x} = f(t)x \ ,$$

and denote

$$\int_{t_0}^{t} f(s)\, ds \quad \text{by} \quad F(t) \ ,$$

we obtain

$$x_0 = 1$$
$$x_1 = 1 + \int_{t_0}^{t} f(s)\, ds = 1 + F(t)$$
$$x_2 = 1 + \int_{t_0}^{t} f(s)\, ds + \int_{t_0}^{t} F(s)f(s)\, ds = 1 + F(t) + \frac{1}{2}\,(F(t))^2$$
$$x_3 = 1 + \int_{t_0}^{t} f(s)\, ds + \int_{t_0}^{t} F(s)f(s)\, ds + \frac{1}{2}\int_{t_0}^{t} F^2(s)f(s)\, ds$$
$$= 1 + F(t) + \frac{1}{2}\,(F(t))^2 + \frac{1}{3!}\,(F(t))^3$$
$$\cdots$$
$$x_n = 1 + \sum_{r=1}^{n} \frac{1}{r!}\,(F(t))^r$$
$$x = \exp(F(t))$$

Unfortunately, matrix multiplication is not commutative in general. This means that if we denote

$$M(t) = \int_{t_0}^{t} A(s)\, ds \ ,$$

then

$$\frac{d}{dt} M^2(t) = A(t)M(t) + M(t)A(t) \neq 2A(t)M(t) \quad \text{in general,}$$

and we cannot express

$$\int_{t_0}^{t} A(s)M(s)\, ds \quad \text{as} \quad \frac{1}{2}M^2(t) \ .$$

We can only do this if we are fortunate enough to have a coefficient matrix $A(t)$ which commutes with its integral. When this happens we can write down

$$\Phi(t, t_0) = I + \sum_{r=1}^{\infty} \frac{1}{r!} M^r(t) = \exp(M(t)) \ .$$

In particular, if $A(t) = f(t)I + g(t)B$, where B is a constant matrix, then $A(t)$ and $M(t)$ commute, and the fundamental solution is

$$\Phi(t, t_0) = e^{F(t)} \exp(G(t)B) \ ,$$

where $F(t) = \int_{t_0}^{t} f(s)\, ds$ and $G(t) = \int_{t_0}^{t} g(s)\, ds$.

Elementary methods.

There are some systems for which fundamental matrices may be found by elementary methods involving linear algebra.

1. The constant coefficient case. A is independent of t.

(Note that in this case the solutions exist for all t.)

Consider the system $\underline{\dot{x}} = A\underline{x}$, where A is a constant matrix.

If we substitute $\underline{x} = e^{\lambda t}\underline{c}$, where $\underline{c}$ is a constant vector, then $\underline{\dot{x}} = \lambda e^{\lambda t}\underline{c}$, and $\underline{x}$ is a solution of $\underline{\dot{x}} = A\underline{x}$ provided $\lambda\underline{c} = A\underline{c}$; that is, provided $(\lambda I - A)\underline{c} = \underline{0}$.

The non-trivial ($\underline{c} \neq \underline{0}$) solutions of this form correspond to $|\lambda I - A| = 0$; that is, λ is an eigenvalue of A, and $\underline{c}$ is the corresponding eigenvector.

Therefore, provided A has n distinct eigenvalues λ_i with corresponding eigenvectors $\underline{c}_i$, the matrix

$$\left(e^{\lambda_1 t}\underline{c}_1 \ , \ e^{\lambda_2 t}\underline{c}_2 \ , \ \ldots \ , \ e^{\lambda_n t}\underline{c}_n\right)$$

is a fundamental matrix for the system.

When there are repeated eigenvalues, it is still possible to find a fundamental matrix, but the process involves the *Jordan Canonical Form*, and becomes more complicated. Straightforward results are available only for systems derived from single n^{th} order linear equations.

Example. Consider

$$\underline{\dot{x}} = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix} \underline{x} \; .$$

The characteristic polynomial is

$$|\lambda I - A| = \begin{vmatrix} \lambda - 3 & 1 & -1 \\ 1 & \lambda - 5 & 1 \\ -1 & 1 & \lambda - 3 \end{vmatrix}$$

$$= (\lambda - 3)(\lambda - 5)(\lambda - 3) - 1 - 1 - (\lambda - 5) - (\lambda - 3) - (\lambda - 3)$$

$$= (\lambda - 3)(\lambda^2 - 8\lambda + 15) - 3(\lambda - 3) = (\lambda - 3)(\lambda^2 - 8\lambda + 12) = (\lambda - 3)(\lambda - 2)(\lambda - 6) \; . \blacksquare$$

The eigenvalues are $2, 3, 6$.

Corresponding to $\lambda = 2$, the eigenvector $(\xi, \eta, \zeta)'$ satisfies

$$\begin{pmatrix} -1 & 1 & -1 \\ 1 & -3 & 1 \\ -1 & 1 & -1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \; ; \; \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = c \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Therefore $(e^{2t}, 0, -e^{2t})'$ is one solution.

Corresponding to $\lambda = 3$, the eigenvector $(\xi, \eta, \zeta)'$ satisfies

$$\begin{pmatrix} 0 & 1 & -1 \\ 1 & -2 & 1 \\ -1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \; ; \; \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = c \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Therefore $(e^{3t}, e^{3t}, e^{3t})'$ is another solution.

Corresponding to $\lambda = 6$, the eigenvector $(\xi, \eta, \zeta)'$ satisfies

$$\begin{pmatrix} 3 & 1 & -1 \\ 1 & 1 & 1 \\ -1 & 1 & 3 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \; ; \; \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = c \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Therefore $(e^{6t}, -2e^{6t}, e^{6t})'$ is a third solution.

The fundamental matrix formed by these solutions is

$$\begin{pmatrix} e^{2t} & e^{3t} & e^{6t} \\ 0 & e^{3t} & -2e^{6t} \\ -e^{2t} & e^{3t} & e^{6t} \end{pmatrix}$$

The determinant of this matrix, which is the Wronskian of the three solutions, is $6e^{11t}$. This agrees with Abel's Identity. The trace of the coefficient matrix is

$$a_{11} + a_{22} + a_{33} = 3 + 5 + 3 = 11 \; ,$$

whose integral is $11t$.

When $t = 0$, this matrix has the value

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{pmatrix},$$

therefore the fundamental matrix which takes the value I when $t = 0$ is

$$\begin{pmatrix} e^{2t} & e^{3t} & e^{6t} \\ 0 & e^{3t} & -2e^{6t} \\ -e^{2t} & e^{3t} & e^{6t} \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ -1 & 1 & 1 \end{pmatrix}^{-1}$$

$$= \begin{pmatrix} e^{2t} & e^{3t} & e^{6t} \\ 0 & e^{3t} & -2e^{6t} \\ -e^{2t} & e^{3t} & e^{6t} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

$$= e^{2t} \begin{pmatrix} \frac{1}{2} & 0 & -\frac{1}{2} \\ 0 & 0 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} + e^{3t} \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{pmatrix} + e^{6t} \begin{pmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix}$$

This form for the answer is called a *Spectral Decomposition.*

Since in this case A and its integral tA obviously commute, this solution represents $\exp(tA)$.

2. Euler equations..

The other case in which there is a simple procedure for finding the fundamental matrix for the system $\dot{\underline{x}} = A(t)\underline{x}$ is when the coefficient matrix $A = (1/t)B$, where B is a constant matrix.

Note that in this case the coefficient matrix is discontinuous at $t = 0$, so that the existence and uniqueness theorems hold on $(-\infty, 0)$ and on $(0, \infty)$, but not on intervals containing the origin.

For $t > 0$, we obtain solutions to the system by substituting $\underline{x} = t^{\lambda}\underline{c}$.. Then $\dot{\underline{x}} = \lambda t^{\lambda-1}\underline{c}$, and $\dot{\underline{x}} = t^{-1}B\underline{x}$ implies $\lambda\underline{c} = B\underline{c}$.

As in the previous case, we obtain non-trivial solutions if λ is an eigenvalue of B and $\underline{c}$ is the corresponding eigenvector. Therefore, provided B has n distinct eigenvalues, we can determine a fundamental solution on $(0, \infty)$.

If $t < 0$, the substitution $\tau = -t$ converts

$$\frac{d\underline{x}}{dt} = \frac{1}{t}B\underline{x} \quad \text{into} \quad \frac{d\underline{x}}{d\tau} = \frac{1}{\tau}B\underline{x}\ ,$$

so that we have the same form for the solutions on $(-\infty, 0)$.

(Some authors write these solutions in terms of $|t|$ for compactness.)

Example. Consider

$$\dot{\underline{x}} = \frac{1}{t}\begin{pmatrix} 1 & 2 & -2 \\ 2 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}\underline{x}\ .$$

The characteristic polynomial is

$$|\lambda I - B| = \begin{vmatrix} \lambda - 1 & -2 & 2 \\ -2 & \lambda & -1 \\ 2 & -1 & \lambda \end{vmatrix}$$

$$= (\lambda - 1)\lambda^2 + 4 + 4 - 4\lambda - (\lambda - 1) - 4\lambda = (\lambda - 1)(\lambda^2 - 9)$$

The eigenvalues are $1, 3, -3$.

Corresponding to $\lambda = 1$, the eigenvector $(\xi, \eta, \zeta)'$ satisfies

$$\begin{pmatrix} 0 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \; ; \; \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = c \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

Therefore $(0, t, t)'$ is a solution.

Corresponding to $\lambda = 3$, the eigenvector $(\xi, \eta, \zeta)'$ satisfies

$$\begin{pmatrix} 2 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \; ; \; \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = c \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$$

Therefore $(2t^3, t^3, -t^3)'$ is a solution.

Corresponding to $\lambda = -3$, the eigenvector $(\xi, \eta, \zeta)'$ satisfies

$$\begin{pmatrix} -4 & -2 & 2 \\ -2 & -3 & -1 \\ 2 & -1 & -3 \end{pmatrix} \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \; ; \; \begin{pmatrix} \xi \\ \eta \\ \zeta \end{pmatrix} = c \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

Therefore $(t^{-3}, -t^{-3}, t^{-3})'$ is a solution.

The corresponding fundamental matrix is

$$\begin{pmatrix} 0 & 2t^3 & t^{-3} \\ t & t^3 & -t^{-3} \\ t & -t^3 & t^{-3} \end{pmatrix} ,$$

and the solution which reduces to I when $t = 1$ is

$$t \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} + t^3 \begin{pmatrix} \frac{2}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{1}{6} & -\frac{1}{6} \\ -\frac{1}{3} & -\frac{1}{6} & \frac{1}{6} \end{pmatrix} + t^{-3} \begin{pmatrix} \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \end{pmatrix} .$$

This is another case in which A and its integral commute. In this case the fundamental matrix is $\exp(B \log t)$ $(= t^B)$.

Inhomogeneous equations.

Consider the general inhomogeneous system $\underline{\dot{x}} = A\underline{x} + \underline{b}$.

If $\underline{X}$ and $\underline{Y}$ are solutions of this equation on the same interval I, then

$$\underline{\dot{X}} - \underline{\dot{Y}} = A\underline{X} + \underline{b} - A\underline{Y} - \underline{b} = A(\underline{X} - \underline{Y}) \; .$$

That is, $\underline{X} - \underline{Y}$ is a solution of the reduced equation $\underline{\dot{x}} = A\underline{x}$. If we know $\underline{Y}$ and a general solution $c_1\underline{Y}_1 + \cdots + c_n\underline{Y}_n$ of the reduced equation, then it follows that

$$\underline{X} = \underline{Y} \quad + \quad c_1\underline{Y}_1 + \cdots + c_n\underline{Y}_n$$

for some choice of $\underline{c}$. Furthermore, for any $t_0 \in I$ and any $\underline{a}$, it is possible to choose the unknowns $\underline{c}$ so that

$$c_1\underline{Y}_1(t_0) + \cdots + c_n\underline{Y}_n(t_0) = \underline{a} - \underline{Y}(t_0)$$

that is, $\underline{X}(t_0) = \underline{a}$, and this form is a general solution of the inhomogeneous system on I.

This means that when solving an inhomogeneous system the process can be broken into two stages:

1. Finding a **complementary function** which is the general solution of the reduced equation.

2. Finding a **particular solution** of the inhomogeneous system.

Furthermore, the second stage can itself be broken up into simpler steps if need be.

If we can write $\underline{b} = \underline{b}_1 + \underline{b}_2$, and if $\dot{\underline{Y}}_{p1} = A\underline{Y}_{p1} + \underline{b}_1$ and $\dot{\underline{Y}}_{p2} = A\underline{Y}_{p2} + \underline{b}_2$ then $\dot{\underline{Y}}_{p1} + \dot{\underline{Y}}_{p2} = A(\underline{Y}_{p1} + \underline{Y}_{p2}) + (\underline{b}_1 + \underline{b}_2)$. That is, we can determine the particular solution piecemeal.

Inspired guesswork.

One approach to finding a particular solution is to guess the form of a solution and then to substitute this form into the equation to confirm the guess. Mostly this can be a hit-and-miss waste of time, but there are two general cases (corresponding to the two special cases, dealt with earlier, for which a fundamental solution can be simply obtained) for which a simple form of substitution is available.

1. The matrix A is a constant, and the vector $\underline{b} = e^{rt}\underline{c}$, where $\underline{c}$ is a constant vector.

We look for a solution of the form $\underline{x} = e^{rt}\underline{d}$, where $\underline{d}$ is a constant vector which needs to be calculated.

Substituting this form into the equation $\dot{\underline{x}} = A\underline{x} + \underline{b}$, we obtain

$$\begin{aligned} re^{rt}\underline{d} &= Ae^{rt}\underline{d} + e^{rt}\underline{c} \\ (rI - A)\underline{d} &= \underline{c} \\ \underline{d} &= (rI - A)^{-1}\underline{c} \end{aligned}$$

provided $|rI - A| \neq 0$, i.e., provided r is not an eigenvalue of A.

e.g. Consider the system

$$\dot{\underline{x}} = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix} \underline{x} + e^t \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} .$$

If $\underline{x} = e^t \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$, then $\begin{pmatrix} -2 & 1 & -1 \\ 1 & -4 & 1 \\ -1 & 1 & -2 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$; $\begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} -\frac{2}{5} \\ -\frac{1}{5} \\ -\frac{2}{5} \end{pmatrix}$.

Therefore a particular solution is $\underline{x}_p = -\frac{1}{5}e^t(2,1,2)'$, and the general solution is

$$\underline{x} = \begin{pmatrix} e^{2t} & e^{3t} & e^{6t} \\ 0 & e^{3t} & -2e^{6t} \\ -e^{2t} & e^{3t} & e^{6t} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} - \frac{1}{5}e^t \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} .$$

2. The matrix $A = (1/t)B$, where B is a constant matrix, and the vector $\underline{b} = t^{r-1}\underline{c}$, where $\underline{c}$ is a constant vector.

We look for a solution of the form $\underline{x} = t^r \underline{d}$, where $\underline{d}$ is a constant vector which needs to be calculated.

Substituting this form into the equation $\dot{\underline{x}} = A\underline{x} + \underline{b}$, we obtain

$$\begin{aligned} rt^{r-1}\underline{d} &= t^{r-1}B\underline{d} + t^{r-1}\underline{c} \\ (rI - B)\underline{d} &= \underline{c} \\ \underline{d} &= (rI - B)^{-1}\underline{c} \end{aligned}$$

provided $|rI - B| \neq 0$, i.e., provided r is not an eigenvalue of B.

e.g. Consider the system

$$t\dot{\underline{x}} = \begin{pmatrix} 1 & 2 & -2 \\ 2 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix} \underline{x} + t^2 \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} .$$

If $\quad \underline{x} = t^2 \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix}$, then $\begin{pmatrix} 1 & -2 & 2 \\ -2 & 2 & -1 \\ 2 & -1 & 2 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} ; \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} .$

A particular solution is $\underline{x} = (-t^2, 0, t^2)'$, and the general solution is

$$\underline{x} = \begin{pmatrix} 0 & 2t^3 & t^{-3} \\ t & t^3 & -t^{-3} \\ t & -t^3 & t^{-3} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} + t^2 \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} .$$

The method of variation of parameters.

There is a general method which can be used to find a particular solution of an inhomogeneous system if the general solution of the reduced system is known.

Suppose that the matrix $\mathcal{X}(t)$ is a fundamental matrix for the homogeneous system $\dot{\underline{x}} = A\underline{x}$. In order to solve the inhomogeneous system $\dot{\underline{x}} = A\underline{x} + \underline{b}$ we make the substitution $\underline{x} = \mathcal{X}(t)\underline{\xi}$. Then

$$\begin{aligned} \dot{\underline{x}} &= \dot{\mathcal{X}}\underline{\xi} + \mathcal{X}\dot{\underline{\xi}} \\ &= A\mathcal{X}\underline{\xi} + \mathcal{X}\dot{\underline{\xi}} \\ &= A\underline{x} + \mathcal{X}\dot{\underline{\xi}} \\ \text{Hence} \qquad \mathcal{X}\dot{\underline{\xi}} &= \underline{b} \\ \dot{\underline{\xi}} &= \mathcal{X}^{-1}(t)\underline{b}(t) \\ \underline{\xi} &= \int_{t_0}^{t} \mathcal{X}^{-1}(s)\underline{b}(s)\, ds , \end{aligned}$$

and the general solution of the inhomogeneous system is

$$\underline{x} = \mathcal{X}(t)\underline{c} + \mathcal{X}(t) \int_{t_0}^{t} \mathcal{X}^{-1}(s)\underline{b}(s)\, ds .$$

Note that, since $\mathcal{X}(t)$ is a fundamental solution of the reduced system, $|\mathcal{X}(t)| \neq 0$, so that the inverse exists.

Example. Consider the inhomogeneous system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} (t+1) & (1-t-t^2) \\ 1 & -t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} e^t \\ t \end{pmatrix} .$$

We know already that

$$\mathcal{X}(t) = \begin{pmatrix} t & (t+1)e^t \\ 1 & e^t \end{pmatrix}$$

is a fundamental matrix for the reduced system, so that we can use it to determine a solution for the inhomogeneous system.

$$|\mathcal{X}(t)| = te^t - (t+1)e^t = -e^t$$

$$\mathcal{X}^{-1}(t) = -e^{-t} \begin{pmatrix} e^t & -(t+1)e^t \\ -1 & t \end{pmatrix} = \begin{pmatrix} -1 & (t+1) \\ e^{-t} & -te^{-t} \end{pmatrix}$$

$$\mathcal{X}^{-1}(t)\underline{b}(t) = \begin{pmatrix} -1 & (t+1) \\ e^{-t} & -te^{-t} \end{pmatrix} \begin{pmatrix} e^t \\ t \end{pmatrix} = \begin{pmatrix} -e^t + t^2 + t \\ 1 - t^2 e^{-t} \end{pmatrix}$$

$$\int_0^t \mathcal{X}^{-1}(s)\underline{b}(s)\, ds = \begin{pmatrix} 1 - e^t + \frac{1}{3}t^3 + \frac{1}{2}t^2 \\ t + (t^2 + 2t + 2)e^{-t} - 2 \end{pmatrix}$$

$$\underline{x} = \mathcal{X}(t)\underline{c} + \begin{pmatrix} t & (t+1)e^t \\ 1 & e^t \end{pmatrix} \begin{pmatrix} 1 - e^t + \frac{1}{3}t^3 + \frac{1}{2}t^2 \\ t + (t^2 + 2t + 2)e^{-t} - 2 \end{pmatrix}$$

$$x_1 = (c_1 + 1)t + (c_2 - 2)(t+1)e^t + t^2 e^t + \frac{1}{4}t^4 + \frac{3}{2}t^3 + 3t^2 + 4t + 2$$

$$x_2 = (c_1 + 1) + (c_2 - 2)e^t + (t-1)e^t + \frac{1}{3}t^3 + \frac{3}{2}t^2 + 2t + 2$$

Two examples from Boyce and DiPrima.

p168 - 5. Determine a particular solution of

$$y'' + y = \tan x, \qquad 0 < x < \pi/2$$

using the method of variation of parameters.

Consider firstly the homogeneous equation $y'' + y = 0$. The characteristic equation is $m^2 + 1 = 0$, the characteristic roots are $m = \pm i$, and the complementary function is $y_c = c_1 \cos x + c_2 \sin x$.

The corresponding system is

$$\frac{d}{dx} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} + \begin{pmatrix} 0 \\ \tan x \end{pmatrix} ,$$

and corresponding to the solution $\cos x$ of the original equation we have the solution $(\cos x, -\sin x)'$ for the system, while corresponding to the solution $\sin x$ of the original equation we have the solution $(\sin x, \cos x)'$ for the system.

We seek the solution of the inhomogeneous equation in the form $y = u(x)\cos x + v(x)\sin x$,which corresponds to setting

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix}$$

in the system. This gives

$$\begin{pmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{pmatrix} \begin{pmatrix} u'(x) \\ v'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ \tan x \end{pmatrix}$$
$$\begin{pmatrix} u'(x) \\ v'(x) \end{pmatrix} = \begin{pmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{pmatrix} \begin{pmatrix} 0 \\ \tan x \end{pmatrix}$$
$$u'(x) = -\sin x \tan x = -\frac{\sin^2 x}{\cos x}$$
$$= \frac{\cos^2 x - 1}{\cos x} = \cos x - \sec x.$$
$$u(x) = \sin x - \log(\sec x + \tan x) + c_1.$$
$$v'(x) = \cos x \tan x = \sin x$$
$$v(x) = c_2 - \cos x$$
$$y = c_1 \cos x + c_2 \sin x + \sin x \cos x - \cos x \log(\sec x + \tan x) - \sin x \cos x$$
$$y = c_1 \cos x + c_2 \sin x - \cos x \log(\sec x + \tan x).$$

p169 - 16. Verify that e^x and x are solutions of the homogeneous equation corresponding to

$$(1-x)y'' + xy' - y = 2(x-1)^2 e^{-x}, \qquad 0 < x < 1,$$

and find the general solution.

If $y_1 = x$, $y_1' = 1$, $y_1'' = 0$, then

$$(1-x)y_1'' + xy_1' - y_1 = (1-x).0 + x.1 - x = 0.$$

Similarly, if $y_2 = e^x$, $y_2' = e^x$, and $y_2'' = e^x$, so that

$$(1-x)y_2'' + xy_2' - y^2 = (1-x)e^x + xe^x - e^x = 0.$$

Therefore, both y_1 and y_2 are solutions of the homogeneous equation for $0 < x < 1$. Since the Wronskian

$$W(x, e^x) = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = (x-1)e^x$$

does not vanish on the designated interval, these solutions are independent on $(0, 1)$.

(In fact, they are independent solutions on both $(-\infty, 1)$ and $(1, \infty)$, but the statement of the problem does not require this.)

We seek a solution of the inhomogeneous equation in the form $y = u(x)x + v(x)e^x$, or

$$\begin{pmatrix} y \\ y' \end{pmatrix} = \begin{pmatrix} x & e^x \\ 1 & e^x \end{pmatrix} \begin{pmatrix} u(x) \\ v(x) \end{pmatrix},$$

if we consider the equivalent system. Making this substitution gives the matrix system

$$\begin{pmatrix} x & e^x \\ 1 & e^x \end{pmatrix} \begin{pmatrix} u'(x) \\ v'(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -2(x-1)e^{-x} \end{pmatrix}$$

$$\begin{pmatrix} u'(x) \\ v'(x) \end{pmatrix} = \frac{1}{(x-1)e^x} \begin{pmatrix} e^x & -e^x \\ -1 & x \end{pmatrix} \begin{pmatrix} 0 \\ -2(x-1)e^{-x} \end{pmatrix}$$

$$u'(x) = \frac{e^x.0 + e^x 2(x-1)e^{-x}}{(x-1)e^x} = 2e^{-x}$$

$$u(x) = c_1 - 2e^{-x}$$

$$v'(x) = \frac{-x2(x-1)e^{-x}}{(x-1)e^x} = -2xe^{-2x}$$

$$v(x) = c_2 + (x + \frac{1}{2})e^{-2x}$$

$$y = c_1 x + c_2 e^x - 2xe^{-x} + (x + \frac{1}{2})e^{-x} = c_1 x + c_2 e^x - (x - \frac{1}{2})e^{-x}.$$

Examples of Exam Questions

1. Find the general solution of the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + e^t \begin{pmatrix} 1 \\ 1 \end{pmatrix} .$$

2. Find the general solution of the system

$$t\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} t \\ t^2 \end{pmatrix} .$$

3. Show that $\underline{y}_1 = (1, -1-t)'$ and $\underline{y}_2 = (e^{-t}, -te^{-t})'$ are independent solutions of the system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -t-1 & -1 \\ t^2+t-1 & t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} ,$$

and hence find the general solution of the inhomogeneous system

$$\frac{d}{dt}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -t-1 & -1 \\ t^2+t-1 & t \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} 0 \\ te^{-t} \end{pmatrix} .$$

4. Define the Wronskian of the two solutions $\underline{\phi}_1(t) = (\phi_{11}(t), \phi_{21}(t))'$ and $\underline{\phi}_2(t) = (\phi_{12}(t), \phi_{22}(t))'$ of the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} ,$$

and derive an expression for its value in terms of the coefficients of the system.

5. Define the term *fundamental matrix* associated with the system

$$\underline{\dot{x}} = A(t)\underline{x} ,$$

and show how it can be used to derive a general solution for the inhomogeneous equation

$$\underline{\dot{x}} = A(t)\underline{x} + \underline{f}(t) .$$