MATH 2000

Tutorial Sheet 2

Solutions

1. Find an LU decomposition for each of the following matrices:

$$(a)\begin{pmatrix}2&4\\0&2\\3&6\end{pmatrix}\quad(b)\begin{pmatrix}2&6&-4\\3&14&-16\\-4&-12&11\end{pmatrix}\quad(c)\begin{pmatrix}2&-1&3&4\\2&0&4&2\\-4&2&-4&-7\\-2&1&-17&0\end{pmatrix}$$

Write down the value of the determinant in each case (if it is defined).

Ans

(a) $$L=\begin{pmatrix}1&0&0\\0&1&0\\\frac{3}{2}&0&1\end{pmatrix}\ ;\ U=\begin{pmatrix}2&4\\0&2\\0&0\end{pmatrix}$$

(b) $$L=\begin{pmatrix}1&0&0\\\frac{3}{2}&1&0\\-2&0&1\end{pmatrix}\ ;\ U=\begin{pmatrix}2&6&-4\\0&5&-10\\0&0&3\end{pmatrix}\ ;\ |A|=30$$

(c) $$L=\begin{pmatrix}1&0&0&0\\1&1&0&0\\-2&0&1&0\\-1&0&-7&1\end{pmatrix}\ ;\ U=\begin{pmatrix}2&-1&3&4\\0&1&1&-2\\0&0&2&1\\0&0&0&11\end{pmatrix}\ ;\ |A|=44$$

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2. Find a PLU decomposition for each of the following matrices:

$$(a)\begin{pmatrix}0&0&0\\2&4&0\end{pmatrix}\quad(b)\begin{pmatrix}1&1&1\\1&1&0\\1&0&0\end{pmatrix}\quad(c)\begin{pmatrix}0&1&3\\2&0&0\\4&0&0\end{pmatrix}\quad(d)\begin{pmatrix}0&2\\2&4\\3&6\end{pmatrix}$$

Write down the value of the determinant in each case (if it is defined).

Ans

(a) $$\begin{pmatrix}0&1\\1&0\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}2&4&0\\0&0&0\end{pmatrix}$$

(b) $$\begin{pmatrix}1&0&0\\0&0&1\\0&1&0\end{pmatrix}\begin{pmatrix}1&0&0\\1&1&0\\1&0&1\end{pmatrix}\begin{pmatrix}1&1&1\\0&-1&-1\\0&0&-1\end{pmatrix}\ ;\ |A|=-1$$

or

$$\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\begin{pmatrix}1&0&0\\1&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

(c) $$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\2&0&1\end{pmatrix}\begin{pmatrix}2&0&0\\0&1&3\\0&0&0\end{pmatrix}\ ;\ |A|=0$$

(d) $$\begin{pmatrix}0&1&0\\1&0&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\\frac{3}{2}&0&1\end{pmatrix}\begin{pmatrix}2&4\\0&2\\0&0\end{pmatrix}$$

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3. Let

$$A = \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

(i) Find the LU decomposition of A.

Ans

$$L = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 & 0 \\ 0 & -\frac{2}{3} & 1 & 0 \\ 0 & 0 & -\frac{3}{4} & 1 \end{pmatrix} \; ; \; U = \begin{pmatrix} 2 & -1 & 0 & 0 \\ 0 & \frac{3}{2} & -1 & 0 \\ 0 & 0 & \frac{4}{3} & -1 \\ 0 & 0 & 0 & \frac{5}{4} \end{pmatrix}$$

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(ii) **Hence** determine the solution of $A\boldsymbol{x} = \boldsymbol{b}$, where $\boldsymbol{b} = (1, 0, 0, 2)'$.

Ans If $L\boldsymbol{y} = \boldsymbol{b}$, then

$$\begin{aligned} y_1 &= 1 \\ -\frac{1}{2}y_1 + y_2 &= 0 \; ; \; y_2 = \frac{1}{2} \\ -\frac{2}{3}y_2 + y_3 &= 0 \; ; \; y_3 = \frac{1}{3} \\ -\frac{3}{4}y_3 + y_4 &= 2 \; ; \; y_4 = \frac{9}{4} \end{aligned}$$

If $U\boldsymbol{x} = \boldsymbol{y}$, then

$$\begin{aligned} \frac{5}{4}x_4 &= \frac{9}{4} \; ; \; x_4 = \frac{9}{5} = 1.8 \\ \frac{4}{3}x_3 = x_4 + \frac{1}{3} &= \frac{32}{15} \; ; \; x_3 = \frac{8}{5} = 1.6 \\ \frac{3}{2}x_2 = x_3 + \frac{1}{2} &= \frac{21}{10} \; ; \; x_2 = \frac{7}{5} = 1.4 \\ 2x_1 = x_2 + 1 &= \frac{12}{5} \; ; \; x_1 = \frac{6}{5} = 1.2 \end{aligned}$$

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(iii) Determine L^{-1} and U^{-1}, and hence calculate A^{-1}.

Ans

$$L^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{1}{3} & \frac{2}{3} & 1 & 0 \\ \frac{1}{4} & \frac{2}{4} & \frac{3}{4} & 1 \end{pmatrix}$$

$$U^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{1}{3} & \frac{1}{4} & \frac{1}{5} \\ 0 & \frac{2}{3} & \frac{2}{4} & \frac{2}{5} \\ 0 & 0 & \frac{3}{4} & \frac{3}{5} \\ 0 & 0 & 0 & \frac{4}{5} \end{pmatrix}$$

$$A^{-1} = U^{-1}L^{-1} = \begin{pmatrix} \frac{4}{5} & \frac{3}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{3}{5} & \frac{6}{5} & \frac{4}{5} & \frac{2}{5} \\ \frac{2}{5} & \frac{4}{5} & \frac{6}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{3}{5} & \frac{4}{5} \end{pmatrix}$$

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(iv) Determine $|A|$.
Ans $|A| = |L||U| = 5$. ■

4. Find an **orthonormal** basis for the solution space of the equation

$$x_1 \;+\; x_2 \;-\; x_3 \;-\; x_4 \;=\; 0\;.$$

Ans We first find a basis for the solution space.
We have

$$x_4 = x_1 + x_2 - x_3$$

so that if (x_1, x_2, x_3, x_4) is a solution it has the form

$$\begin{aligned} &(x_1, x_2, x_3, x_1 + x_2 - x_3) \\ &= x_1(1,0,0,1) + x_2(0,1,0,1) + x_3(0,0,1,-1) \end{aligned}$$

Therefore $\{(1,0,0,1),(0,1,0,1),(0,0,1,-1)\}$ is a basis for the solution space.
We now apply the Gram-Schmidt process to this set.

$$\begin{aligned} ||(1,0,0,1)||^2 &= 1^2 + 0^2 + 0^2 + 1^2 = 2 \\ \boldsymbol{e}_1 &= \left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right) \\ \langle (0,1,0,1), \boldsymbol{e}_1 \rangle &= 0 + 0 + 0 + \frac{1}{\sqrt{2}} \\ \boldsymbol{y}_2 &= (0,1,0,1) - \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}, 0, 0, \frac{1}{\sqrt{2}}\right) \\ &= \left(-\frac{1}{2}, 1, 0, \frac{1}{2}\right) \\ ||\boldsymbol{y}_2||^2 &= \frac{1}{4} + 1 + 0 + \frac{1}{4} = \frac{3}{2} \\ \boldsymbol{e}_2 &= \left(-\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}\right) \\ \langle (0,0,1,-1), \boldsymbol{e}_1 \rangle &= 0 + 0 + 0 - \frac{1}{\sqrt{2}} \\ \langle (0,0,1,-1), \boldsymbol{e}_2 \rangle &= 0 + 0 + 0 - \frac{1}{\sqrt{6}} \\ \boldsymbol{y}_3 &= (0,0,1,-1) + \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right) + \left(-\frac{1}{6}, \frac{1}{3}, 0, \frac{1}{6}\right) \\ &= \left(\frac{1}{3}, \frac{1}{3}, 1, -\frac{1}{3}\right) \\ ||\boldsymbol{y}_3||^2 &= \frac{1}{9} + \frac{1}{9} + 1 + \frac{1}{9} = \frac{4}{3} \\ \boldsymbol{e}_3 &= \left(\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, \frac{\sqrt{3}}{2}, -\frac{1}{2\sqrt{3}}\right) \end{aligned}$$

The vectors $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ are an orthonormal basis for the solution space.

This basis is not unique. It depends on the original choice of basis for the solution space and on the ordering of the vectors in this basis.

If you have a different set of vectors, you should check that your vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ are indeed solutions of the original equation, and that they are mutually orthogonal.