Hermitian Operators

Suppose that A is a real $n \times n$ matrix.
While the characteristic polynomial

$$|\lambda I - A|$$

has real coefficients, there is no guarantee that the roots of the characteristic equation will be real.

For example, if

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is the matrix of a transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$,

$$\begin{aligned} |\lambda I - A| &= \begin{vmatrix} \lambda & -1 \\ 1 & \lambda \end{vmatrix} \\ &= \lambda^2 + 1 \end{aligned}$$

so that the roots of the characteristic equation are $\lambda = \pm i$.

The Adjoint Operator

If V is an inner product space, and T is a linear transformation on V, we define the adjoint of T, denoted T^* by the rule

$$\langle T^* \boldsymbol{y}, \boldsymbol{x} \rangle = \langle \boldsymbol{y}, T\boldsymbol{x} \rangle$$

If $V = \mathbb{C}^n$, and the matrix of T is A, then

$$\begin{aligned} \langle \boldsymbol{y}, \boldsymbol{x} \rangle &= \overline{\boldsymbol{y}}^t \boldsymbol{x} \\ \langle \boldsymbol{y}, T\boldsymbol{x} \rangle &= \overline{\boldsymbol{y}}^t (A\boldsymbol{x}) \\ &= \overline{(\overline{A}^t \boldsymbol{y})}^t \boldsymbol{x} \end{aligned}$$

so that the matrix of T^* is the conjugate transpose of A.

For economy, we usually write

$$\overline{A}^t = A^* \ .$$

When we are working in real vector spaces, we can dispense with the conjugate.

Of special interest are operators with the property that

$$T^* = T$$

Such an operator is said to be *Hermitian* or *self-adjoint*.

We have three crucial results:

1. If T is a self-adjoint operator on V, and $\boldsymbol{x}$ is any vector in V, then $\langle \boldsymbol{x}, T\boldsymbol{x} \rangle$ is a real number.

Proof:

$$\langle \boldsymbol{x}, T\boldsymbol{x} \rangle = \langle T^* \boldsymbol{x}, \boldsymbol{x} \rangle = \langle T\boldsymbol{x}, \boldsymbol{x} \rangle = \overline{\langle \boldsymbol{x}, T\boldsymbol{x} \rangle}$$

2. The eigenvalues of a Hermitian operator are real.

Proof: If

$$T\boldsymbol{x} = \lambda\boldsymbol{x} \ , \ \boldsymbol{x} \neq \boldsymbol{0}$$

then

$$\begin{aligned} \langle \boldsymbol{x}, T\boldsymbol{x}\rangle &= \lambda\langle \boldsymbol{x}, \boldsymbol{x}\rangle \\ \lambda &= \langle \boldsymbol{x}, T\boldsymbol{x}\rangle / \langle \boldsymbol{x}, \boldsymbol{x}\rangle \end{aligned}$$

which is real.

3. If $\boldsymbol{x}$ and $\boldsymbol{y}$ are eigenvalues of a Hermitian operator, belonging to distinct eigenvalues λ and ν respectively, then $\boldsymbol{x}$ and $\boldsymbol{y}$ are orthogonal.

Proof:

$$\begin{aligned} \text{Since} \quad & T\boldsymbol{x} = \lambda\boldsymbol{x} \ , \ \langle \boldsymbol{y}, T\boldsymbol{x}\rangle = \lambda\langle \boldsymbol{y}, \boldsymbol{x}\rangle \\ \text{Since} \quad & T\boldsymbol{y} = \nu\boldsymbol{y} \ , \ \langle \boldsymbol{x}, T\boldsymbol{y}\rangle = \nu\langle \boldsymbol{x}, \boldsymbol{y}\rangle \end{aligned}$$

Hence

$$\begin{aligned} \nu\langle \boldsymbol{x}, \boldsymbol{y}\rangle &= \langle \boldsymbol{x}, T\boldsymbol{y}\rangle \\ &= \langle T^*\boldsymbol{x}, \boldsymbol{y}\rangle \\ &= \langle T\boldsymbol{x}, \boldsymbol{y}\rangle \\ &= \overline{\lambda}\langle \boldsymbol{x}, \boldsymbol{y}\rangle \\ &= \lambda\langle \boldsymbol{x}, \boldsymbol{y}\rangle \\ (\nu - \lambda)\langle \boldsymbol{x}, \boldsymbol{y}\rangle &= 0 \\ \langle \boldsymbol{x}, \boldsymbol{y}\rangle &= 0 \end{aligned}$$

as required.

For example, if V is the space of real polynomials, with the inner product

$$\langle p(x), q(x)\rangle = \int_0^\infty e^{-x} p(x) q(x) \, dx \ ,$$

then the transformation

$$T(p(x)) = x\frac{d^2p}{dx^2} + (1-x)\frac{dp}{dx}$$

on V is self-adjoint.

$$
\begin{aligned}
\langle p(x), T(q(x))\rangle &= \int_0^\infty xe^{-x}p(x)\frac{d^2q}{dx^2}\,dx \\
&\quad + \int_0^\infty (1-x)e^{-x}p(x)\frac{dq}{dx}\,dx \\
&= xe^{-x}p(x)\frac{dq}{dx}\bigg|_0^\infty \\
&\quad - \int_0^\infty (1-x)e^{-x}p(x)\frac{dq}{dx}\,dx \\
&\quad - \int_0^\infty xe^{-x}\frac{dp}{dx}\,\frac{dq}{dx}\,dx \\
&\quad + \int_0^\infty (1-x)e^{-x}p(x)\frac{dq}{dx}\,dx \\
&= -\int_0^\infty xe^{-x}\frac{dp}{dx}\,\frac{dq}{dx}\,dx \\
\langle T(p(x)), q(x)\rangle &= -\int_0^\infty xe^{-x}\frac{dp}{dx}\,\frac{dq}{dx}\,dx \\
&= \langle p(x), T(q(x))\rangle
\end{aligned}
$$

We have already seen that the eigenvalues of this operator are $0, -1, -2, \dots$, and that the eigenvectors are the Laguerre polynomials, which are orthogonal with respect to this inner product.

While we have assumed that we may be working in a general complex valued vector space in order to derive these results, the main application is in $\mathbb{R}^n$.

In this case the matrices involved are *symmetric*: $A^t = A$.

The important result is that the eigenvalues of a real symmetric matrix are real. Therefore, in this case we do not have to worry about possible complex conjugate eigenvalues.

We also have a complete set of orthonormal eigenvectors, so that a symmetric matrix is always diagonalisable.

Where the eigenvalues are distinct, the eigenvectors are automatically orthogonal. Where we have repeated eigenvalues, we need to use the Gram-Schmidt process to produce orthonormal eigenvectors.

For example, let

$$
A = \begin{pmatrix} 2 & -2 & 1 \\ -2 & 5 & -2 \\ 1 & -2 & 2 \end{pmatrix}
$$

$$\begin{aligned}
|tI - A| &= \begin{vmatrix} t-2 & 2 & -1 \\ 2 & t-5 & 2 \\ -1 & 2 & t-2 \end{vmatrix} \\
&= (t-2)^2(t-5) - 4 - 4 \\
&\quad - (t-5) - 4(t-2) - 4(t-2) \\
&= t^3 - 4t^2 + 4t - 5t^2 + 20t - 20 - 8 - 9t + 21 \\
&= t^3 - 9t^2 + 15t - 7 \\
&- (t-1)^2(t-7)
\end{aligned}$$

The eigenvalues are 7, 1, 1 (which are real).

Corresponding to $\lambda = 7$, the eigenvector $(x, y, z)'$ satisfies

$$\begin{aligned}
5x + 2y - z &= 0 \\
2x + 2y + 2z &= 0 \\
-x + 2y + 5z &= 0
\end{aligned}$$

$$\begin{aligned}
3x - 3z &= 0 \\
x = z\ ;\ y &= -2z \\
(x, y, z) &= z(1, -2, 1)
\end{aligned}$$

$$||(1, -2, 1)||^2 = 1^2 + (-2)^2 + 1^2 = 6$$

so that the normalised eigenvector is

$$\left(\frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)'$$

Corresponding to $\lambda = 1$, the eigenvectors $(x, y, z)'$ satisfy

$$\begin{aligned}
-x + 2y - z &= 0 \\
2x - 4y + 2z &= 0 \\
-x + 2y - z &= 0
\end{aligned}$$

$$\begin{aligned}
x &= 2y - z \\
(x, y, z) = (2y - z, y, z) &= y(2, 1, 0) + z(-1, 0, 1)
\end{aligned}$$

To find orthonormal eigenvectors, we need to use the Gram-Schmidt process.

$$||(2, 1, 0)||^2 = 2^2 + 1^2 + 0 = 5$$

so that the first orthonormal eigenvector ($\boldsymbol{e}_1$) is

$$\left(\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}, 0\right)'$$

$$\langle(-1,0,1), \boldsymbol{e}_1\rangle = -\frac{2}{\sqrt{5}}$$

so that an orthogonal eigenvector is

$$(-1,01) + (\frac{4}{5}, \frac{2}{5}, 0) = (-\frac{1}{5}, \frac{2}{5}, 1)$$

$$||(-\frac{1}{5}, \frac{2}{5}, 1)||^2 = \frac{1}{25} + \frac{4}{25} + 1 = \frac{30}{25}$$

so that the second orthonormal eigenvector is

$$\left(-\frac{1}{\sqrt{30}}, \frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}\right)'$$

Note that this choice of orthonormal basis for the eigenspace is not unique. For example, applying the Gram-Schmidt process to the original basis in the reverse order (*exercise for students*) gives a different orthonormal basis.