CYCLES FOR A GENERALIZED COLLATZ PROBLEM

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In 1937 L. Collatz introduced the mapping $T : \mathbb{Z} \to \mathbb{Z}$ given by

$$T(x) = \begin{cases} (3x+1)/2 & \text{if } x \text{ is odd} \\ x/2 & \text{if } x \text{ is even} \end{cases}$$

It is believed that if $x \in \mathbb{Z}^+$ then the trajectory

$$x,\ T(x),\ T^2(x),\ \ldots$$

eventually collapses onto the cycle $\{2, 1\}$.

Similarly, for $x \in \mathbb{Z}^-$ it is believed that the trajectory $\{T^n(x)\}$ eventually collapses onto one of the cycles

$$\{-1\}$$
$$\{-5, -7, -10\}$$
$$\{-17, -25, -37, -55, -82, -41, -61, -91, -136, -68, -34\}$$

This mapping has since been generalized in a variety of ways. The generalization with which this note is concerned replaces the mapping by

$$T_k(x) = \begin{cases} (3x+k)/2 & \text{if } x \text{ is odd} \\ x/2 & \text{if } x \text{ is even} \end{cases}$$

where k is a positive odd integer; $3 \nmid k$.

This generalization introduces a richer family of cycles. For convenience we characterize cycles by the odd integer of least magnitude in the cycle. The cycles for the Collatz mapping can therefore be characterized by [1], [−1], [−5] and [−17].

It should be noted that if the g.c.d. $(x, k) = d$, then $(T^n(x), k) = d$ also.

Therefore, corresponding to the trajectory $\{T_k^n(x)\}$, there is a trajectory $\{T_{k/d}^n(\frac{x}{d})\}$, and these trajectories are isomorphic.

Cycles of this type will be said to be *inherited*, while cycles for which $(x, k) = 1$ will be called *primitive*.

Also, if $k = k_1k_2$, and $[x]$ is a cycle belonging to the mapping T_{k_1}, then $[k_2x]$ is an isomorphic cycle belonging to T_k.

In particular, for every k, T_k has the cycles $[k]$, $[-k]$, $[-5k]$ and $[-17k]$.

Cycles with one odd element

If x is the only odd element of the cycle $[x]$, then

$$3x + k = 2^n x$$

where $n \geq 1$ is the length of the cycle.

This gives

$$k = (2^n - 3)x$$

When $n = 1$ this gives the cycle $[-k]$, and for $n = 2$ we have the cycle $[k]$. These cycles are inherited by all mappings T_k.

Otherwise we require that k be a multiple of $2^n - 3$. In particular, when $k = 2^n - 3$ we have the primitive cycle [1] of length n. This shows trivially that given any cycle length n we can find k such that T_k has a cycle of length n. However, in practice T_k will have cycles of much greater length, and $2^n - 3$ merely provides a crude upper bound for the occurrence of a cycle of length n.

For example, for $n = 8$, $k = 253$, in addition to the cycle [1] of length 8, T_k has primitive cycles [13] of length 162 and [17] of length 42. It also has the cycles inherited from $k = 1$, $k = 11$ and $k = 23$.

Cycles with two odd elements

If the cycle has two odd elements, x and y, then

$$3x + k = 2^m y$$
$$3y + k = 2^n x$$

where $n > m \geq 1$, and the cycle $[x]$ has length $n + m$.

Eliminating k, we have

$$(2^n + 3)x = (2^m + 3)y$$

An obvious solution of this equation is

$$x = 2^m + 3$$
$$y = 2^n + 3$$

for which

$$k = 2^{m+n} - 9 \ .$$

Therefore, when $k = 2^{2N+1} - 9$ there are N distinct cycles of length $2N + 1$ for T_k corresponding to $m = 1, \ldots N$, while when $k = 2^{2N} - 9$ there are $N - 1$ such cycles of length $2N$, the case $m = n = N$ giving a cycle of length N inherited from $k = 2^N - 3$.

We can trivially obtain other solutions of

$$(2^n + 3)x = (2^m + 3)y$$

by setting

$$x = r(2^m + 3)$$
$$y = r(2^n + 3)$$
$$k = r(2^{m+n} - 9)$$

There are also simpler solutions for x and y if $(2^n+3, 2^m+3) = d > 1$, namely

$$\begin{aligned} x &= (2^m+3)/d \\ y &= (2^n+3)/d \\ k &= (2^{m+n}-9)/d \ . \end{aligned}$$

However, in this case the cycle generated by x is an ancestor of the corresponding cycle listed above, and is isomorphic to it. Hence when $k = 2^M - 9$, the set of cycles belonging to k contains isomorphic copies of all distinct cycles of length M which contain two odd elements, and it is the smallest value of k with this property.

General Cycles

Suppose that T_k has a cycle $[x]$ containing t distinct odd elements x_i, x_2, $\dots, x_t$. Then for some positive integers n_1, n_2, $\dots, n_t$, we have

$$\begin{aligned} 3x_1 + k &= 2^{n_1} x_2 \\ 3x_2 + k &= 2^{n_2} x_3 \\ &\dots\dots \\ 3x_t + k &= 2^{n_t} x_1 \end{aligned}$$

which represents a cycle of length $N = n_1 + n_2 + \cdots + n_t$.

We can represent this as

$$A\boldsymbol{x} = \begin{pmatrix} 3 & -2^{n_1} & 0 & \dots & 0 \\ 0 & 3 & -2^{n_2} & \dots & 0 \\ . & . & . & \dots & . \\ -2^{n_t} & 0 & 0 & \dots & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ . \\ x_t \end{pmatrix} = -k \begin{pmatrix} 1 \\ 1 \\ . \\ 1 \end{pmatrix}$$

Cramer's Rule gives

$$|A| x_i = -|A_i| k \ ; \ i = 1, \dots, t$$

where $|A| = 3^t - 2^{n_1+n+2+\cdots+n_t} = 3^t - 2^N$, and A_i is the array derived from A by replacing the i^{th} column of A by $(1, 1, \dots, 1)'$.

Specifically,

$$\begin{aligned} |A_1| &= \begin{vmatrix} 1 & -2^{n_1} & 0 & \dots & 0 \\ 1 & 3 & -2^{n_2} & \dots & 0 \\ . & . & . & \dots & . \\ 1 & 0 & 0 & \dots & 3 \end{vmatrix} \\ &= \begin{vmatrix} 3 & -2^{n_2} & \dots & 0 \\ 0 & 3 & \dots & 0 \\ . & . & \dots & . \\ 0 & 0 & \dots & 3 \end{vmatrix} \\ &\quad + \ 2^{n_1} \begin{vmatrix} 1 & -2^{n_2} & 0 & \dots & 0 \\ 1 & 3 & -2^{n_3} & \dots & 0 \\ . & . & . & \dots & . \\ 1 & 0 & 0 & \dots & 3 \end{vmatrix} \\ &= 3^{t-1} + 2^{n_1} \left(3^{t-2} + 2^{n_2} \left(\cdots + (3 + 2^{n_{t-1}}) \dots \right) \right) \end{aligned}$$

with the other values $|A_i|$ having a similar evaluation with a cyclic permutation of the indices n_j.

In particular, if $n_1 = n_2 = \cdots = n_{t-1} = 1$,

$$|A_1| = 3^t - 2^t \ .$$

If $3^t - 2^N > 0$, we can take $k = 3^t - 2^N$ and $x_i = -|A_i|$, $i = 1, \ldots, t$, as a generic example of such a cycle, while if $3^t - 2^N < 0$, we can take $k = 2^N - 3^t$ and $x_i = |A_i|$. Note that k is odd and not divisible by 3, and $|A_i|$ is a positive odd integer.

In either case, for these choices of k, T_k has cycles of length N with t odd elements corresponding to all appropriate choices of n_i. If, for some choice of $n_1, n_2, \ldots, n_t$, $(|A|, |A_1|) = d > 1$ then $(|A|, |A_i|) = d$ for each i, and we can construct an isomorphic primitive cycle corresponding to this selection with $T_{|A|/d}$ and the odd elements $\{|A_i|/d\}$.

However, $k = |3^t - 2^N|$ is the least value of k for which all possible such cycles occur.