THE **z**-TRANSFORM

Definition.

Given a sequence $\{a_0, a_1, a_2, \dots\}$, we define the z-Transform of this sequence as

$$\mathfrak{z}(a_n) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots = \sum_{n=0}^{\infty} \frac{a_n}{z^n} \ ,$$

provided this series converges.

Using the Root test, the series converges provided

$$\text{limsup} \left| \frac{a_n}{z^n} \right|^{1/n} < 1$$

that is, provided

$$|z| > \text{limsup}\, |a_n|^{1/n} \ .$$

Alternatively, provided the limit exists we can use the Ratio test, which gives convergence provided

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n z} \right| < 1 \ ; \quad \text{i.e.} \quad |z| > \lim \left| \frac{a_{n+1}}{a_n} \right| \ .$$

Both these tests give the same value R for the region $|z| > R$ in which the series converges.

Note that this is a series in inverse powers of z, so that it is the Laurent expansion of a function regular in a neighbourhood of infinity.

Where the sequence is generated by periodic sampling of a function $f(t)$, we have $a_n = f(nT)$, where T is the sampling period. In this case the transform is usually written as $\mathfrak{z}(f(t))$, where the sampling period is implied.

Examples.

1. The constant sequence: $a_n = c$.

$$\mathfrak{z}(c) = \sum_{n=0}^{\infty} \frac{c}{z^n} = \frac{c}{1 - z^{-1}} = \frac{cz}{z-1} \ .$$

In this case the transform converges for $|z| > 1$ to a function which is regular except at $z = 1$.

2. The geometric sequence: $a_n = cr^n$.

$$\mathfrak{z}(cr^n) = \sum_{n=0}^{\infty} c \left(\frac{r}{z} \right)^n = \frac{c}{1 - r/z} = \frac{cz}{z - r} \ .$$

In this case the transform converges for $|z| > |r|$, and the transform function is regular except at $z = r$. (Note: r may be complex.)

3. The arithmetic sequence: $a_n = a + bn$.

$$\begin{aligned}
\mathfrak{z}(a+bn) &= \sum_{n=0}^{\infty} \frac{a+bn}{z^n} \\
&= \sum_{n=0}^{\infty} \frac{a}{z^n} + \sum_{n=0}^{\infty} \frac{bn}{z^n} \\
&= \frac{a}{1-z^{-1}} + \frac{bz^{-1}}{(1-z^{-1})^2} \\
&= \frac{az}{z-1} + \frac{bz}{(z-1)^2}\ .
\end{aligned}$$

In this case the transform has a pole of order 2 at $z = 1$.

4. The quadratic sequence: $a_n = n^2$.

Consider the following.

$$\begin{aligned}
\text{If} \qquad \phi(t) &= 1 + t + t^2 + \cdots = \sum t^n \\
\text{then} \qquad \phi'(t) &= 1 + 2t + 3t^2 + \cdots = \sum nt^{n-1} \\
t\phi'(t) &= t + 2t^2 + 3t^3 + \cdots = \sum nt^n \\
(t\phi'(t))' &= 1 + 4t + 9t^2 + \cdots = \sum n^2 t^{n-1} \\
t(t\phi'(t))' &= t + 4t^2 + 9t^3 + \cdots = \sum n^2 t^n \\
\text{Since} \quad \phi(t) &= \frac{1}{1-t}\ , \ \sum n^2 t^n = \frac{t+t^2}{(1-t)^3}
\end{aligned}$$

From this it follows that

$$\mathfrak{z}(n^2) = \frac{z^2+z}{(z-1)^3}\ .$$

Properties.

1. The General Inversion Formula

If $\mathfrak{z}(a_n)$ converges to $F(z)$ for $|z| > R$, then, for any $R_1 > R$, Laurent's Theorem gives

$$a_n = \frac{1}{2\pi i} \oint_{|z|=R_1} F(z) z^{n-1}\, dz\ .$$

Since $F(z)$ is regular for $|z| > R$, if the function F is single valued, this integral can be evaluated as the sum of the residues of $F(z)z^{n-1}$ in the finite plane.

e.g. If $F(z) = z/(z-1)(z-2)$, then

$$\begin{aligned}
a_n &= \operatorname{Res} \left.\frac{z^n}{(z-1)(z-2)}\right|_{z=1} + \operatorname{Res} \left.\frac{z^n}{(z-1)(z-2)}\right|_{z=2} \\
&= -1 + 2^n
\end{aligned}$$

2.Linearity
If $\mathfrak{z}(a_n) = F_1(z)$ and $\mathfrak{z}(b_n) = F_2(z)$, then

$$\mathfrak{z}(\alpha a_n + \beta b_n) = \alpha F_1(z) + \beta F_2(z) \ .$$

e.g. We can use partial fractions to write the previous example as

$$F(z) = \frac{z}{z-2} - \frac{z}{z-1}$$

from which the result follows from the earlier examples.

3. Shifting theorem
If $\mathfrak{z}(a_n) = F(z)$, then, for $r \geq 1$,

$$\begin{aligned} \mathfrak{z}(a_{n-r}) &= z^{-r} F(z) \\ \mathfrak{z}(a_{n+r}) &= z^{r} \left(F(z) - \sum_{k=0}^{r-1} a_k z^{-k} \right) \end{aligned}$$

e.g.

$$\mathfrak{z}(2^{n+1} - 1) = z \left(\frac{z}{(z-1)(z-2)} - (2^0 - 1) \right) = \frac{z^2}{(z-1)(z-2)}$$

4. Scaling
If $\mathfrak{z}(a_n) = F(z)$, then $\mathfrak{z}(a_n r^n) = F(z/r)$.
e.g.

$$\begin{aligned} \mathfrak{z}(nr^n) &= \frac{z/r}{(z/r-1)^2} = \frac{zr}{(z-r)^2} \\ \mathfrak{z}(nr^{n-1}) &= \frac{z}{(z-r)^2} \end{aligned}$$

5. Convolution
If $\mathfrak{z}(a_n) = F_1(z)$ and $\mathfrak{z}(b_n) = F_2(z)$, then

$$F_1(z).F_2(z) = \mathfrak{z} \left(\sum_{k=0}^{n} a_k b_{n-k} \right) \ .$$

e.g.

$$\mathfrak{z}(1) = \frac{z}{z-1} \qquad \mathfrak{z}(2^n) = \frac{z}{z-2}$$

$$\begin{aligned} \text{therefore} \quad \frac{z^2}{(z-1)(z-2)} &= \mathfrak{z} \left(\sum_{k=0}^{n} 2^k \right) \\ &= \mathfrak{z}(2^{n+1} - 1) \end{aligned}$$

6. The Hadamard product

If $\mathfrak{z}(a_n) = F_1(z)$ for $|z| > R_1$, and $\mathfrak{z}(b_n) = F_2(z)$ for $|z| > R_2$, then

$$\mathfrak{z}(a_n b_n) = \frac{1}{2\pi i} \oint \frac{F_1(\zeta) F_2(z/\zeta)}{\zeta}\, d\zeta \ ,$$

where the contour is chosen so that $|\zeta| > R_1$ and $|z/\zeta| > R_2$.

Proof:

$$\begin{aligned}
\mathfrak{z}(a_n b_n) &= \sum_{n=0}^{\infty} a_n b_n z^{-n} \\
&= \sum_{n=0}^{\infty} \left(\frac{1}{2\pi i} \oint F_1(\zeta) \zeta^{n-1}\, d\zeta \right) b_n z^{-n} \\
&= \frac{1}{2\pi i} \oint \frac{F_1(\zeta)}{\zeta} \left(\sum_{n=0}^{\infty} b_n \zeta^n z^{-n} \right) d\zeta \\
&= \frac{1}{2\pi i} \oint \frac{F_1(\zeta) F_2(z/\zeta)}{\zeta}\, d\zeta
\end{aligned}$$

where the inversion formula requires $|\zeta| > R_1$ and the convergence of the infinite sum requires $|z/\zeta| > R_2$. Note that these together imply that $|z| > R_1 R_2$, which ensures convergence.

e.g. $\mathfrak{z}(n) = z/(z-1)^2$ for $|z| > 1$, therefore

$$\begin{aligned}
\mathfrak{z}(n^2) &= \frac{1}{2\pi i} \oint \frac{\zeta}{(\zeta-1)^2} \frac{z/\zeta}{(z/\zeta-1)^2} \frac{d\zeta}{\zeta} \qquad 1 < |\zeta| < |z| \\
&= \frac{1}{2\pi i} \oint \frac{z\zeta}{(\zeta-1)^2 (z-\zeta)^2}\, d\zeta \\
&= \text{the residue at the pole } \zeta = 1 \\
&= \frac{d}{d\zeta} \left. \frac{z\zeta}{(z-\zeta)^2} \right|_{\zeta=1} \\
&= \frac{z}{(z-1)^2} + \frac{2z}{(z-1)^3}
\end{aligned}$$

Applications.

The z-Transform is designed to be applied to difference equations with constant coefficients.

Consider the problem:

Determine the solution of the difference equation

$$x_{n+2} + c_1 x_{n+1} + c_2 x_n = d_n \ ; \ x_0 = \alpha \ , \ x_1 = \beta$$

where c_1 and c_2 are constants, and $\{d_n\}$ is a known sequence.

Applying the z-Transform, we obtain

$$\mathfrak{z}(x_{n+2}) + c_1 \mathfrak{z}(x_{n+1}) + c_2 \mathfrak{z}(x_n) = \mathfrak{z}(d_n) \ .$$

Denoting $\mathfrak{z}(x_n) = \mathcal{X}(z)$ and $\mathfrak{z}(d_n) = \mathcal{D}(z)$, we have

$$\begin{aligned} z^2 \left(\mathcal{X}(z) - \alpha - \frac{\beta}{z} \right) + c_1 z(\mathcal{X}(z) - \alpha) + c_2 \mathcal{X}(z) &= \mathcal{D}(z) \\ (z^2 + c_1 z + c_2)\mathcal{X}(z) &= \alpha z^2 + (\beta + c_1 \alpha) z + \mathcal{D}(z) \\ \mathcal{X}(z) &= \frac{\alpha z^2 + (\beta + c_1\alpha) z}{z^2 + c_1 z + c_2} + \frac{\mathcal{D}(z)}{z^2 + c_1 z + c_2} \end{aligned}$$

If r_1 and r_2 are the (distinct) roots of $z^2 + c_1 z + c_2 = 0$, then the first term on the right hand side can be written as

$$\frac{\alpha r_1 + \beta + c_1 \alpha}{r_1 - r_2} \frac{z}{z - r_1} + \frac{\alpha r_2 + \beta + c_1 \alpha}{r_2 - r_1} \frac{z}{z - r_2}$$

which is the z-Transform of

$$\frac{\beta - \alpha r_2}{r_1 - r_2} \, r_1^n + \frac{\beta - \alpha r_1}{r_2 - r_1} \, r_2^n$$

This *complementary function* represents the solution of the homogeneous problem.

If $r_1 = r_2$, this term can be written as

$$\alpha \frac{z}{z - r_1} + (\beta - r_1 \alpha) \frac{z}{(z - r_1)^2}$$

which is the z-Transform of

$$\alpha r_1^n + (\beta - r_1 \alpha) n r_1^{n-1} \ .$$

The determination of the particular solution depends on the form of $\mathcal{D}$.

TUTORIAL SHEET 10

1. Determine the z-Transforms of

(a) $$a_n = \sin(n\omega)$$

(b) $$b_n = nr^{n-1}\sin(n\omega)$$

2. Find the solution of the difference equation

$$x_{n+2} + x_n = \sin(n) \qquad x_0 = 1\ ,\ x_1 = 0\ .$$

3. If

$$x_{n+2} - 3x_{n+1} + 2x_n = u_{n+1} + u_n \qquad x_0 = x_1 = 0\ ;$$

determine x_n in terms of $u_0, \ldots, u_{n-1}$.

Assignment: Question **2**.