Convergence

Let

$$\sum_{n=1}^{\infty} z_n = z_1 + \cdots + z_n + \ldots$$

be an infinite series with complex terms z_n.

This series converges if and only if the sequence

$$\{s_n\} = \{z_1 + \cdots + z_n\}$$

of partial sums converges.

If

$$\lim_{n\to\infty} s_n = s = \sigma + i\tau$$

then we write

$$\sum_{n=1}^{\infty} z_n = s$$

and if $z_n = x_n + iy_n$,

$$\sum_{n=1}^{\infty} x_n = \sigma \ ; \ \sum_{n=1}^{\infty} y_n = \tau$$

The sequence $\{s_n\}$ converges if and only if, given any $\epsilon > 0$, there is an integer $N(\epsilon)$ such that

$$|s_m - s_n| < \epsilon \ \forall \ m, n > N \ .$$

Hence the series converges if and only if, given any $\epsilon > 0$, there is an integer $N(\epsilon)$ such that

$$|z_{n+1} + \cdots + z_{n+p}| < \epsilon \ \forall \ n > N \text{ and } p > 0 \ .$$

In particular, a *NECESSARY* condition for the series to converge is that

$$\lim_{n\to\infty} z_n = 0 \ .$$

Absolute convergence

The series is said to be *absolutely convergent* if the series

$$\sum_{n=1}^{\infty} |z_n| = |z_1| + \cdots + |z_n| + \ldots$$

converges.

Since

$$|z_{n+1} + \cdots + z_{n+p}| \leq |z_{n+1}| + \cdots + |z_{n+p}|$$

an absolutely convergent series converges.

The converse does not hold.

The series

$$\sum_{n=1}^{\infty} |z_n|$$

is a sum of non-negative real numbers.

Therefore, the partial sums

$$S_n = \sum_{i=1}^{n} |z_i|$$

associated with this series form a monotonic increasing sequence in $\mathbb{R}$.

Such a sequence converges if and only if it is bounded.

The most common means of proving the absolute convergence of a sequence is by means of *comparison*.

If we have a sequence of positive real numbers a_n such that

$$\sum_{n=1}^{\infty} a_n$$

converges,

and if, for some N

$$|z_n| \leq a_n \ \forall \ n > N$$

then

$$\sum_{n=1}^{\infty} z_n$$

converges absolutely.

The most common series used for comparison is the Geometric progression.

The ratio test

If

$$\lim_{n\to\infty} \frac{|z_{n+1}|}{|z_n|} = l < 1$$

then

$$\sum_{n=1}^{\infty} z_n$$

converges absolutely.

If the limit is > 1 then the series diverges, while if $l = 1$ or the limit does not exist, then the test is inconclusive.

Proof: Suppose $l < 1$.

Let $\epsilon = \frac{1}{2}(1 - l)$. The exists $N(\epsilon)$ such that for $n \geq N$,

$$\left| \frac{|z_{n+1}|}{|z_n|} - l \right| < \epsilon$$

$$\frac{|z_{n+1}|}{|z_n|} < l + \epsilon = \frac{1}{2}(l + 1) = r < 1$$

$$|z_{N+p}| \leq |z_N| r^p$$

Since

$$\sum_{p=0}^{\infty} |z_N| r^p$$

converges,

$$\sum_{n=0}^{\infty} z_n$$

converges absolutely.

Conversely, if $l > 1$, we set $\epsilon = \frac{1}{2}(l-1)$.

This gives

$$\frac{|z_{n+1}|}{|z_n|} > l - \epsilon = \frac{1}{2}(l+1) = r > 1$$

Therefore $|z_n| \to \infty$ as $n \to \infty$, and the series diverges.

The Cauchy root test

(Interlude): Given a sequence $\{a_n\}$, The number l is a *limit point* of the sequence if, given any $\epsilon > 0$ there are infinitely many members of the sequence for which $|a_n - l| < \epsilon$.

In other words, we can choose a subsequence $\{a_{n_i}\}$ which converges to l.

The least upper bound of the limit points of the sequence is denoted the lim sup of the sequence. Denote this by l.

Since there is no larger limit point for the sequence, given any $\epsilon > 0$ there are only a finite number of terms of the sequence greater than $l + \epsilon$.

Equally, if there were only a finite number of terms of the series greater than $l - \epsilon$ there would be no limit point greater than $l - \epsilon$. Therefore since l is the lub of the limit points, there are infinitely many points of the sequence greater than $l - \epsilon$.

If the sequence converges, the lim sup is the lim.

Cauchy's Root test

Let $\limsup_{n\to\infty} |z_n|^{1/n} = l$.

If $l < 1$, the series converges absolutely.

If $l > 1$, the series diverges.

If $l = 1$, the test is inconclusive.

Proof: If $l < 1$, then, given $\epsilon = \frac{1}{2}(1-l)$, there is an $N(\epsilon)$ such that for $n > N$

$$|z_n|^{1/n} < l + \epsilon = \frac{1}{2}(1+l) = r < 1$$
$$|z_n| < r^n$$

so that the series converges absolutely.

Conversely, if $l > 1$, take $\epsilon = \frac{1}{2}(l-1)$. Then there are infinitely many terms such that

$$|z_n|^{1/n} > l - \epsilon = \frac{1}{2}(l+1) = r > 1$$
$$|z_n| > r^n$$

Since these terms do not go to 0 as $n \to \infty$, the series diverges.

Uniform convergence

The sequence of functions $f_n(z)$ converge uniformly to the function $f(z)$ on a set S if given any $\epsilon > 0$ there is a $N(\epsilon, S) > 0$ such that

$$|f(z) - f_n(z)| < \epsilon \ \forall \ n > N \text{ AND } \forall \ z \in S$$

Similarly, the series

$$\sum_{n=0}^{\infty} f_n(z)$$

converges uniformly to $f(z)$ on S if the sequence of partial sums converges uniformly to $f(z)$.

Note that if a sequence converges uniformly on S, it converges uniformly on any subset of S.

Theorem: If the sequence of continuous functions $f_n(z)$ converge uniformly on S to $f(z)$, then $f(z)$ is continuous on S.
Proof: For z, z_0 in S,

$$\begin{aligned} |f(z) - f(z_0)| &= |(f(z) - f_n(z)) + (f_n(z) - f_n(z_0)) + (f_n(z_0) - f(z_0))| \\ &\le |f(z) - f_n(z)| + |f_n(z) - f_n(z_0)| + |f_n(z_0) - f(z_0)| \end{aligned}$$

Since the convergence is uniform, given $\epsilon > 0$ we can choose N such that

$$|f(z) - f_n(z)| < \frac{1}{3}\epsilon \ \forall \ n > N \ , \ \ z, z_0 \in S$$

Choose such a f_n. Since it is continous on S, given this same ϵ there is $\delta > 0$ such that

$$|f_n(z) - f_n(z_0)| < \frac{1}{3}\epsilon \ \forall \ |z - z_0| < \delta$$

Combining these results we obtain

$$|f(z) - f(z_0)| < \frac{1}{3}\epsilon + \frac{1}{3}\epsilon + \frac{1}{3}\epsilon = \epsilon \ \forall \ |z - z_0| < \delta \ .$$

Therefore f is continuous at every point of S.

Theorem: Given a rectifiable curve C, suppose that the series of continuous functions $f_n(z)$ converge uniformly on C to $f(z)$.
Then

$$\lim_{n\to\infty} \int_C f_n(z)\, dz = \int_C f(z)\, dz$$

Proof:

$$\begin{aligned} \left| \int_C f(z)\, dz - \int_C f_n(z)\, dz \right| &= \left| \int_C (f(z) - f_n(z))\, dz \right| \\ &\le \max_{z \in C} |f(z) - f_n(z)|\, L \end{aligned}$$

where L is the length of C.

Since the convergence is uniform, given any $\epsilon > 0$ we can find N such that $|f(z) - f_n(z)| < \epsilon/L$ for all $n > N$ and for all $z \in C$.

Therefore

$$\left| \int_C f(z)\, dz - \int_C f_n(z)\, dz \right| < \epsilon \ \forall \ n > N$$

as required.

Theorem: If the sequence of regular functions $f_n(z)$ converge uniformly on a simply connected set S to the function $f(z)$, then $f(z)$ is regular on S.
Proof: For any scroc C lying in S

$$\oint_C f(z)\,dz = \lim_{n\to\infty} \oint_C f_n(z)\,dz = 0\ .$$

Therefore, by Morera's theorem $f(z)$ is regular on S.

Theorem: If the sequence of regular functions $f_n(z)$ converge uniformly to $f(z)$ on S, then for every interior point of $z_0 \in S$, $f_n'(z_0)$ converges to $f'(z_0)$.
Proof: If z_0 is an interior point of S, there is an $r > 0$ such that the disc $|z - z_0| \le r$ lies in S. Let C be the circle $|z - z_0| = r$.

Then

$$\begin{aligned} f'(z_0) &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^2}\,dz \\ f_n(z_0) &= \frac{1}{2\pi i}\oint_C \frac{f_n(z)}{(z-z_0)^2}\,dz \\ |f'(z_0) - f_n(z_0)| &= \frac{1}{2\pi}\left|\oint_C \frac{f(z)-f_n(z)}{(z-z_0)^2}\,dz\right| \\ &\le \end{aligned}$$

$$frac12\pi\frac{2\pi r}{r^2}\max|f(z) - f_n(z)|$$

Given $\epsilon > 0$ we can find N such that

$$|f(z) - f_n(z)| < \frac{r\epsilon}{}\forall\ n > N \text{ and } \forall\ z \in C$$

Therefore

$$|f'(z_0) - f_n'(z_0)| < \epsilon\ \forall\ n > N$$

as required.

WEIERSTRASS' M-TEST

Supppose that the functions $f_n(z)$ satisfy the inequalities

$$|f_n(z)| \le M_n$$

for all z in some set S.

If the series

$$\sum_{n=1}^{\infty} M_n$$

converges, then the series

$$\sum_{n=1}^{\infty} f_n(z)$$

converges absolutely and uniformly on S.

Proof: Given any $\epsilon > 0$, there is N such that

$$|M_{n+1} + \cdots + M_{n+p}| < \epsilon \ \forall\ n > N \text{ and } p > 0$$

Since the $M_n \geq 0$,

$$|M_{n+1} + \cdots + M_{n+p}| = M_{n+1} + \cdots + M_{n+p}$$

Therefore, for all $n > N$ and $p > 0$, and for all $z \in S$,

$$\begin{aligned}|f_{n+1}(z) + \cdots + f_{n+p}(z)| &\leq |f_{n+1}(z)| + \cdots + |f_{n+p}(z)| \\ &\leq M_{n+1} + \cdots + M_{n+p} < \epsilon\end{aligned}$$

as required.

For example, if $S = \{z; |z| \leq r < 1\}$, then for each n

$$|z^n| = |z|^n \leq r^n \ .$$

Since $\sum_{n=0}^{\infty} r^n$ converges, $\sum_{n=0}^{\infty} z^n$ converges uniformly to $1/(1-z)$ for $|z| \leq r < 1$.

As we shall see shortly, this behaviour is typical of power series,

Power series

A power series is an expression of the form

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where the a_n are complex valued constants.

Applying the Cauchy root test, we see that the series converges if

$$\begin{gathered}\limsup |a_n (z - z_0)^n|^{1/n} < 1 \\ |z - z_0| \limsup |a_n|^{1/n} < 1 \\ |z - z_0| < 1/(\limsup |a_n|^{1/n}) = R\end{gathered}$$

and similarly, it diverges when

$$|z - z_0| > R$$

When $|z - z_0| = R$, it may or may not converge.

This number R is known as the *radius of convergence* of the power series.

When $R = 0$, the series converges only at $z = z_0$, and is of no further interest in this course. Henceforth we will assume that $R > 0$.

A function represented by such a non-trivial power series expansion is said to be *analytic*.

Since

$$\begin{gathered}\lim_{n \to \infty} n^{1/n} = 1 \\ \limsup |n a_n|^{1/n} = \limsup |a_n|^{1/n}\end{gathered}$$

and the derived series

$$\sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1}$$

has the same radius of convergence as the original series.

Uniform convergence

Suppose that the series $\sum a_n(z-z_0)^n$ converges for $|z-z_0|<R$.
For any $r<R$, the series converges absolutely for $z=z_0+r$.
That is, $\sum |a_n|r^n$ converges.
If $|z-z_0|\le r$, $|a_n(z-z_0)^n|\le |a_n|r^n$, so that by the Weierstrass M-test the power series converges uniformly for $|z-z_0|\le r$.

Since each of the partial sums

$$\sum_{n=0}^{N} a_n(z-z_0)^n$$

is a polynomial, the partial sums are regular functions.
It follows that the power series

$$\sum_{n=0}^{\infty} a_n(z-z_0)^n$$

is a regular function on every set

$$S=\{|z-z_0|\le r<R\}$$

Since for any value of $|z-z_0|<R$ we can choose $r=\frac{1}{2}(R+|z-z_0|)$, the power series represents a regular function for $|z-z_0|<R$.
In short, an analytic function is regular.

Furthermore, for every ζ in $|z-z_0|\le r$,

$$\frac{d}{dz}\sum_{n=0}^{N} a_n(z-z_0)^n\bigg|_{z=\zeta}=\sum_{n=1}^{N} na_n(\zeta-z_0)^{n-1}$$

It follows that

$$\frac{d}{dz}\sum_{n=0}^{\infty} a_n(z-z_0)^n\bigg|_{z=\zeta}=\sum_{n=1}^{\infty} na_n(\zeta-z_0)^{n-1}$$

That is, we obtain the derivative of a power series by differentiating term by term, and the derivative has the same radius of convergence as the original power series.