

MATH 3401
EXTRA TUTORIAL SHEET
SOLUTIONS

1. A rational function $f(z)$ satisfies the conditions;

$$\begin{aligned}f(z) &= 0 \text{ iff } z = \pm i \\f(z) &= 1 \text{ iff } z = 0, 1 \\f(z) &= -4 \text{ iff } z = -2 \pm i\end{aligned}$$

Determine $f(z)$.

Ans Since the function is $2 - 1$, it is the ratio of polynomials of degree 2.

Writing

$$f(z) = \frac{a_2 z^2 + a_1 z + a_0}{b_2 z^2 + b_1 z + b_0}$$

the first condition tells us that the numerator is

$$a_2(z^2 + 1),$$

and without loss of generality we may take $a_2 = 1$.

When $f(z) = 1$,

$$z^2 + 1 = b_2 z^2 + b_1 z + b_0$$

Substituting $z = 0$ and $z = 1$, we obtain

$$\begin{aligned}b_0 &= 1 \\b_2 + b_1 + b_0 &= 2 \\b_2 &= 1 - b_1\end{aligned}$$

When $f(z) = -4$,

$$z^2 + 1 = -4(1 - b_1)z^2 - 4b_1z - 4$$

Substituting $z = -2 \pm i$, we obtain

$$\begin{aligned}3 - 4i + 1 &= -4(1 - b_1)(3 - 4i) - 4b_1(-2 + i) - 4 \\b_1(12 - 16i + 8 - 4i) &= 4 - 4i + 12 - 16i + 4 \\b_1 &= 1 \\b_2 &= 0 \\f(z) &= \frac{z^2 + 1}{z + 1}\end{aligned}$$

2. A function $f(z)$ is said to be *meromorphic* if it is the ratio of entire functions. Show that the singularities of a meromorphic function in the finite complex plane are isolated poles.

Ans The function has its singularities at the zeros of the denominator.

Since the denominator is an entire function, its zeros are of finite order and isolated.

Therefore the singularities are isolated poles.

Under which conditions is the point at infinity

- (a) a removable singularity;
- (b) a pole;
- (c) an essential singularity.

Ans Writing the function as

$$f(z) = \frac{g(z)}{h(z)}$$

where g and h are entire and have no common factors, the function has an essential singularity at ∞ if and only if the equation

$$g(z) = ah(z)$$

has infinitely many solutions for almost all values of a .

Therefore the point at infinity is an essential singularity if and only if g or h or both are transcendental functions.

If both g and h are polynomials, then ∞ is a removable singularity if f is bounded as $z \rightarrow \infty$.

This corresponds to $\partial(g) \leq \partial(h)$.

If $\partial(g) > \partial(h)$ then f has a pole of order $\partial(g) - \partial(h)$ at infinity.

3. If

$$w = \frac{z-1}{z^3},$$

determine z in the form

$$z = \sum_{n=0}^{\infty} a_n w^n.$$

For which values of w does this series converge?

Determine a similar expansion for z^2 .

Ans Using Lagrange's expansion, we have

$$\begin{aligned} z &= 1 + wz^3 \\ &= 1 + \sum_{n=1}^{\infty} \frac{w^n}{n!} D^{n-1}(z^{3n})|_{z=1} \\ &= 1 + \sum_{n=1}^{\infty} \frac{(3n)!}{n!(2n+1)!} w^n \\ z^2 &= 1^2 + \sum_{n=1}^{\infty} \frac{w^n}{n!} D^{n-1}(2z \times z^{3n})|_{z=1} \\ &= 1 + \sum_{n=1}^{\infty} \frac{2(3n+1)!}{n!(2n+2)!} w^n \end{aligned}$$

By the ratio test, the first series converges when

$$\lim_{n \rightarrow \infty} \frac{(3n+3)!n!(2n+1)!}{(n+1)!(2n+3)!(3n)!} |w| < 1$$

$$|w| < \lim_{n \rightarrow \infty} \frac{(n+1)(2n+2)(2n+3)}{(3n+1)(3n+2)(3n+3)} = \frac{4}{27}$$

4. The sequence $\{x_n\}_{n=0}^{\infty}$ satisfies the recurrence relation

$$\begin{aligned} x_{n+2} - 3x_{n+1} + 2x_n &= 2n + 1, \quad n \geq 0 \\ x_0 &= 0 \\ x_1 &= -2 \end{aligned}$$

Determine x_n for $n \geq 2$.

Ans Let $\mathcal{X}(z)$ be the z transform of $\{x_n\}$.

Then

$$\begin{aligned} z^2\left(\mathcal{X} + \frac{2}{z}\right) - 3z\mathcal{X} + 2\mathcal{X} &= \frac{2z}{(z-1)^2} + \frac{z}{z-1} \\ (z^2 - 3z + 2)\mathcal{X} &= \frac{z^2 + z}{(z-1)^2} - 2z \\ (z-2)(z-1)\mathcal{X} &= z \frac{-2z^2 + 5z - 1}{(z-1)^2} \\ \mathcal{X} &= z \frac{-2z^2 + 5z - 1}{(z-2)(z-1)^3} \\ &= \frac{z}{z-2} - \frac{z^3 + z^2}{(z-1)^3} \end{aligned}$$

$$\begin{aligned} x_n &= 2^n - \frac{1}{2\pi i} \oint \frac{z^{n+2} + z^{n+1}}{(z-1)^3} dz \\ &= 2^n - \frac{1}{2} \left((n+2)(n+1)z^n + (n+1)nz^{n-1} \right) \Big|_{z=1} \\ &= 2^n - (n+1)^2 \end{aligned}$$