MATH 3401

Tutorial sheet 9

Solutions

1.

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2}\, dx = \mathrm{Im}\left(\int_{-\infty}^{\infty} \frac{x e^{ix}}{1+x^2}\, dx \right)$$

Therefore consider

$$\oint_C \frac{z e^{iz}}{1+z^2}\, dz$$

over a contour consisting of the real axis from $-R$ to R, and a semicircle of radius R in the upper half plane.

The value of the integral is $2\pi i$ times the residue at $z = i$, namely

$$2\pi i \left. \frac{z e^{iz}}{2z} \right|_{z=i} = \pi i e^{-1} \ .$$

As $R \to \infty$, the value of the integral along the semicircular path goes to 0, (Jordan's Lemma), so that

$$\int_{-\infty}^{\infty} \frac{x e^{ix}}{1+x^2}\, dx = \frac{\pi}{e} i$$

$$\int_{-\infty}^{\infty} \frac{x \sin x}{1+x^2}\, dx = \frac{\pi}{e} \ .$$

2.

$$2 \sum_{n=1}^{\infty} \frac{1}{n^4} = -Res \left. \frac{\pi \cot(\pi z)}{z^4} \right|_{z=0}$$

$$\frac{\pi \cot(\pi z)}{z^4} = \frac{\pi}{z^4} \frac{1 - \frac{1}{2}\pi^2 z^2 + \frac{1}{24}\pi^4 z^4 - \ldots}{\pi z - \frac{1}{6}\pi^3 z^3 + \frac{1}{120}\pi^5 z^5 - \ldots}$$

$$= \frac{1}{z^5}\left(1 - \frac{1}{2}\pi^2 z^2 + \frac{1}{24}\pi^4 z^4 - \ldots\right)\left(1 + \left(\frac{1}{6}\pi^2 z^2 - \frac{1}{120}\pi^4 z^4 + \ldots\right) + \left(\frac{1}{6}\pi^2 z^2 - \ldots\right)^2 + \ldots\right)$$

$$= \frac{1}{z^5}\left(1 - \frac{1}{2}\pi^2 z^2 + \frac{1}{24}\pi^4 z^4 + \frac{1}{6}\pi^2 z^2 - \frac{1}{120}\pi^4 z^4 - \frac{1}{12}\pi^4 z^4 + \frac{1}{36}\pi^4 z^4 + \ldots\right)$$

The residue is

$$\pi^4 \left(\frac{1}{24} - \frac{1}{12} - \frac{1}{120} + \frac{1}{36}\right) = -\frac{\pi^4}{45}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90}$$

3. Setting
$$U(x,p)=\int_0^\infty u(x,t)e^{-pt}\,dt$$
we have
$$U_{xx}=p^2U-pu(x,0)-u_t(x,0)=p^2U$$
where

$$U(0,p)=\int_0^\infty u(0,t)e^{-pt}\,dt=0$$
$$U(\pi,p)=\int_0^\infty u(\pi,t)e^{-pt}\,dt=\frac{1}{p^2}$$

The solution of this boundary value problem is

$$U(x,p)=\frac{1}{p^2}\frac{\sinh(px)}{\sinh(p\pi)}$$

so that
$$u(x,t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{1}{p^2}\frac{\sinh(px)}{\sinh(p\pi)}e^{pt}\,dp$$

Using a Bromwich contour of radius $|p|=N+\frac{1}{2}$ to avoid the poles on the imaginary axis, we have

$$\frac{1}{2\pi i}\oint\frac{1}{p^2}\frac{\sinh(px)}{\sinh(p\pi)}e^{pt}\,dp=\sum\text{ residues inside the contour}$$

There are simple poles at $p=\pm ni$, $n=1,2,\dots,N$, and a pole of order 2 at $p=0$.

At the simple poles the residue is given by

$$\left.\frac{\sinh(px)e^{pt}}{p^2}\,\frac{1}{\pi\cosh(p\pi)}\right|_{p=\pm ni}=\frac{\pm i\sin(nx)e^{\pm int}}{-n^2}\,\frac{1}{\pi\cos(n\pi)}$$

so that their contribution to the integral is

$$\sum_{n=1}^{N}(-1)^{n+1}\frac{\sin(nx)}{\pi n^2}i\left(e^{int}-e^{-int}\right)=\frac{2}{\pi}\sum_{n=1}^{N}(-1)^n\frac{1}{n^2}\sin(nx)\sin(nt)$$

Near $p=0$, the integrand is

$$\frac{1}{p^2}\,\frac{px+\frac{1}{6}p^3x^3+\dots}{p\pi+\frac{1}{6}p^3\pi^3+\dots}(1+pt+\frac{1}{2}p^2t^2+\dots)$$
$$=\frac{1}{p^2}\frac{x}{\pi}(1+pt)+\dots=\frac{x}{\pi p^2}+\frac{xt}{\pi p}+\dots$$

Therefore the residue at $p=0$ is xt/π.

Taking the limit as $N \to \infty$ we obtain

$$u(x,t) = \frac{xt}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin(nx)\sin(nt)\ .$$

4.

$$\mathcal{F}(\omega) = \int_{-\infty}^{\infty}\frac{xe^{-i\omega x}}{x^2+1}\,dx$$

For $\omega < 0$, we can consider a Jordan contour closed in the upper half plane. This gives

$$\int_{-\infty}^{\infty}\frac{xe^{-i\omega x}}{x^2+1}\,dx = 2\pi i\,Res\left(\frac{ze^{-i\omega z}}{z^2+1}\right)\Bigg|_{z=i} = \pi i e^{\omega}$$

For $\omega > 0$, we can consider a Jordan contour closed in the lower half plane. This gives

$$\int_{-\infty}^{\infty}\frac{xe^{-i\omega x}}{x^2+1}\,dx = -2\pi i\,Res\left(\frac{ze^{-i\omega z}}{z^2+1}\right)\Bigg|_{z=-i} = -\pi i e^{-\omega}$$