MATH 3401

Tutorial Sheet 8

Solutions

1. Show that

$$\sum_{n=1}^{\infty} \frac{1}{(n^2+1)^2} = \frac{\pi}{4}\coth\pi + \frac{\pi^2}{4}\operatorname{csch}^2\pi - \frac{1}{2}\ .$$

Ans: We have

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n^2+1)^2} = -\sum \operatorname{Res} \frac{\pi\cot\pi z}{(z^2+1)^2}\bigg|_{z=\pm i}$$

The residue at $z = i$ is

$$\begin{aligned} &\lim_{z\to i} \frac{d}{dz}\frac{\pi\cot\pi z}{(z+i)^2} \\ &= \lim_{z\to i} -\frac{\pi^2\operatorname{cosec}^2\pi z}{(z+i)^2} - 2\frac{\pi\cot\pi z}{(z+i)^3} \\ &= -\frac{\pi^2\operatorname{csch}^2\pi}{4} - \frac{\pi\coth\pi}{4} \end{aligned}$$

Similarly, the residue at $z = -i$ is

$$-\frac{\pi^2\operatorname{csch}^2\pi}{4} - \frac{\pi\coth\pi}{4}$$

Therefore

$$\begin{aligned} 1 + 2\sum_{n=1}^{\infty} \frac{1}{(n^2+1)^2} &= \frac{\pi^2\operatorname{csch}^2\pi}{2} + \frac{\pi\coth\pi}{2} \\ \sum_{n=1}^{\infty} \frac{1}{(n^2+1)^2} &= \frac{\pi^2\operatorname{csch}^2\pi}{4} + \frac{\pi\coth\pi}{4} - \frac{1}{2} \end{aligned}$$

2. Show that

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^4+n^2+1} = \frac{\pi\sqrt{3}}{3} \tanh\left(\frac{\pi\sqrt{3}}{2}\right) .$$

Ans: The function $1/(z^4+z^2+1) = (z^2-1)/(z^6-1)$ has poles at $z = e^{\pm\pi i/3}$ and $z = e^{\pm 2\pi i/3}$.

At $z = e^{\pi i/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$,

$$\begin{aligned}
\cos\pi z &= \cos\left(\frac{\pi}{2} + i\frac{\sqrt{3}\pi}{2}\right) \\
&= -\sin\left(i\frac{\sqrt{3}\pi}{2}\right) \;=\; -i\sinh\left(\frac{\sqrt{3}\pi}{2}\right) \\
\sin\pi z &= \sin\left(\frac{\pi}{2} + i\frac{\sqrt{3}\pi}{2}\right) \\
&= \cos\left(i\frac{\sqrt{3}\pi}{2}\right) \;=\; \cosh\left(\frac{\sqrt{3}\pi}{2}\right) \\
\pi\cot\pi z &= -\pi i\tanh\left(\frac{\sqrt{3}\pi}{2}\right) \\
4z^3+2z &= -4+1+i\sqrt{3} = -3+i\sqrt{3} \\
\frac{1}{4z^3+2z} &= \frac{-3-i\sqrt{3}}{9+3} = -\frac{1}{4} - i\frac{\sqrt{3}}{12} \\
\frac{\pi\cot\pi z}{4z^3+2z} &= -\left(\frac{\pi\sqrt{3}}{12} - i\frac{\pi}{4}\right)\tanh\left(\frac{\sqrt{3}\pi}{2}\right)
\end{aligned}$$

Similarly, at $z = e^{-\pi i/3}$,

$$\frac{\pi\cot\pi z}{4z^3+2z} = -\left(\frac{\pi\sqrt{3}}{12} + i\frac{\pi}{4}\right)\tanh\left(\frac{\sqrt{3}\pi}{2}\right)$$

at $z = e^{2\pi i/3}$,

$$\frac{\pi\cot\pi z}{4z^3+2z} = -\left(\frac{\pi\sqrt{3}}{12} + i\frac{\pi}{4}\right)\tanh\left(\frac{\sqrt{3}\pi}{2}\right)$$

and at $z = e^{-2\pi i/3}$,

$$\frac{\pi\cot\pi z}{4z^3+2z} = -\left(\frac{\pi\sqrt{3}}{12} - i\frac{\pi}{4}\right)\tanh\left(\frac{\sqrt{3}\pi}{2}\right)$$

Since the doubly infinite sum is - the sum of these residues,

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^4+n^2+1} = \frac{\pi\sqrt{3}}{3} \tanh\left(\frac{\pi\sqrt{3}}{2}\right) .$$

3. Show that the function

$$f(z) = \frac{1}{e^z - 1}$$

is bounded on the squares C_N with vertices $\pm(2N+1)\pi \pm (2N+1)\pi i$.

Hence determine the Cauchy partial fraction expansion of this function.

Ans: On the lines $y = \pm(2N+1)\pi$,

$$e^z = e^{x+iy} = e^x e^{\pm(2N+1)\pi i} = -e^x$$

Therefore

$$\left|\frac{1}{e^z - 1}\right| = \frac{1}{e^x + 1} < 1$$

Otherwise,

$$\begin{aligned}
|e^z - 1|^2 &= (e^x \cos y - 1)^2 + (e^x \sin y)^2 \\
&= e^{2x} - 2e^x \cos y + 1 \\
&\geq e^{2x} - 2e^x + 1 \\
\left|\frac{1}{e^z - 1}\right| &\leq \frac{1}{|e^x - 1|} \\
&\leq \frac{1}{1 - e^{-\pi}} \text{ for } |x| \geq \pi
\end{aligned}$$

Therefore $|f| \leq 1/(1 - e^{-\pi}) \sim 1.045$ on C_N.

The function $f(z)$ has simple poles when $e^z = 1$, $z = 2n\pi i$, at which the residues are 1.

Since one of the poles is at the origin, we must consider

$$\begin{aligned}
h(z) &= \frac{1}{e^z - 1} - \frac{1}{z} \\
h(0) &= \lim_{z \to 0} \frac{z + 1 - e^z}{z(e^z - 1)} = -\frac{1}{2}
\end{aligned}$$

Then

$$\begin{aligned}
h(z) &= -\frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{1}{z - 2n\pi i} + \frac{1}{2n\pi i} + \frac{1}{z + 2n\pi i} - \frac{1}{2n\pi i} \right) \\
&= -\frac{1}{2} + \sum_{n=1}^{\infty} \frac{2z}{z^2 + 4n^2\pi^2} \\
f(z) &= \frac{1}{z} - \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2z}{z^2 + 4n^2\pi^2}
\end{aligned}$$

4. Use the representation

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! \exp\left(-\gamma z + \sum_{k=1}^{n} \frac{z}{k}\right)}{z(z+1)\dots(z+n)}$$

to show that

$$\Gamma\left(\frac{1}{2}\right)\Gamma(2p) = 2^{2p-1}\Gamma(p)\Gamma\left(p+\frac{1}{2}\right)$$

Ans:

$$\begin{aligned}
&2^{2p-1}\Gamma(p)\Gamma\left(p+\frac{1}{2}\right) \\
&= \lim_{n\to\infty} 2^{2p-1} \frac{(n!)^2 \exp(-\gamma p - \gamma(p+\frac{1}{2}) + \sum_{k=1}^{n} \frac{1}{k}(p+(p+\frac{1}{2}))}{p(p+1)\dots(p+n)(p+\frac{1}{2})\dots(p+n+\frac{1}{2})} \\
&= \lim_{n\to\infty} 2^{2p-1} \frac{(n!)^2 \exp(-\gamma(2p) - \gamma\frac{1}{2} + \sum_{k=1}^{n} \frac{1}{k}((2p)+\frac{1}{2})}{2^{-2n-2}(2p)(2p+1)\dots(2p+2n+1)} \\
&= \Gamma\left(\frac{1}{2}\right) \lim_{n\to\infty} 2^{2p+2n+1} \frac{n!\frac{1}{2}\frac{3}{2}\dots(n+\frac{1}{2}) \exp(-2p\gamma + (\sum_{k=1}^{2n+1} - \sum_{n+1}^{2n+1} \frac{2p}{k}))}{(2p)(2p+1)\dots(2p+2n+1)} \\
&= \Gamma\left(\frac{1}{2}\right) \lim_{n\to\infty} \frac{(2n+1)! \exp(-\gamma(2p) + \sum_{k=1}^{2n+1} \frac{2p}{k})}{(2p)(2p+1)\dots(2p+2n+1)} \lim_{n\to\infty} \exp\left(2p\log 2 - \sum_{k=n+1}^{2n+1} \frac{2p}{k}\right) \\
&= \Gamma\left(\frac{1}{2}\right)\Gamma(2p) \exp\left(2p\left(\lim_{n\to\infty} \log 2 - \sum_{k=n+1}^{2n+1} \frac{1}{k}\right)\right) \\
&= \Gamma\left(\frac{1}{2}\right)\Gamma(2p)
\end{aligned}$$