MATH 3401

Tutorial Sheet 7

Solutions and Answers

1.

(a)
$$\int_{-\pi}^{\pi} \frac{\cos\theta\, d\theta}{5+4\cos\theta} = \oint_{|z|=1} \frac{\frac{1}{2}(z+z^{-1})}{5+2(z+z^{-1})}\,\frac{dz}{iz}$$
$$= \frac{1}{i}\oint_{|z|=1} \frac{z^2+1}{z(4z^2+10z+4)}\,dz$$
$$= 2\pi\left(\frac{1}{4} + \left.\frac{z^2+1}{z(8z+10)}\right|_{z=-\frac{1}{2}}\right)$$
$$= 2\pi\left(\frac{1}{4}-\frac{5}{12}\right) = -\frac{\pi}{3}$$

$$\int_0^{2\pi} \frac{d\theta}{1+k\sin\theta} = \oint_{|z|=1} \frac{1}{1+\frac{k}{2i}(z-z^{-}1)}\,\frac{dz}{iz}$$
$$= \oint_{|z|=1} \frac{2dz}{kz^2+2iz-k}$$
$$= 2\pi i\,\left.\frac{2}{2kz+2i}\right|_{2kz=-2i+i\sqrt{4-4k^2}}$$
$$= \frac{2\pi}{\sqrt{1-k^2}}$$

(c)
$$\int_0^{\pi} \frac{d\theta}{(a+\cos\theta)^2} = \frac{1}{2}\int_{-\pi}^{\pi} \frac{d\theta}{(a+\cos\theta)^2}$$
$$= \frac{1}{2}\oint_{|z|=1} \frac{4z^2}{(z^2+2az+1)^2}\,\frac{dz}{iz}$$
$$= \pi Res\left(\frac{4z}{(z^2+2az+1)^2}\right)$$
$$= \pi \lim_{z\to -a+\sqrt{a^2-1}} \frac{d}{dz}\left(\frac{4z}{(z+a+\sqrt{a^2-1})^2}\right)$$
$$= \pi\left(\frac{4}{4(a^2-1)} - \frac{8(-a+\sqrt{a^2-1})}{8(a^2-1)^{3/2}}\right)$$
$$= \frac{\pi a}{(a^2-1)^{3/2}}$$

2.

(a)
$$\int_0^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)} = \frac{1}{2}\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)(x^2+4)}$$
$$= \pi i \sum Res\left.\left(\frac{z^2}{(z^2+1)(z^2+4)}\right)\right|_{y>0}$$
$$= (\pi i)\left(\frac{i^2}{2i(i^2+4)} + \frac{(2i)^2}{((2i)^2+1)4i}\right)$$
$$= -\frac{\pi}{6}+\frac{\pi}{3} = \frac{\pi}{6}$$

(b)
$$\int_0^\infty \frac{x^2\,dx}{x^6+1} = \frac{1}{2}\int_{-\infty}^\infty \frac{x^2\,dx}{x^6+1}$$
$$= \pi i \sum Res\left(\frac{z^2}{z^6+1}\right)\Big|_{y>0}$$
$$= \pi i \sum \frac{z_0^2}{6z_0^5}\Big|_{z_0^6=-1}$$
$$= \frac{\pi i}{6}\left(\frac{1}{i}+\frac{1}{-i}+\frac{1}{i}\right) = \frac{\pi}{6}$$

$$\int_0^\infty \frac{x}{x^4+4}\,dx = -\sum Res\left(\frac{z\log z}{z^4+4}\right)_{0<arg(z)<2\pi}$$
$$= -\sum\left(\frac{\log z}{4z^2}\right)_{z=z_i}$$
$$= -\left(\frac{1}{8i}(\frac{1}{2}\log 2 + i\frac{\pi}{4})\right.$$
$$- \frac{1}{8i}(\frac{1}{2}\log 2 + i\frac{3\pi}{4})$$
$$\frac{1}{8i}(\frac{1}{2}\log 2 + i\frac{5\pi}{4})$$
$$\left.-\frac{1}{8i}(\frac{1}{2}\log 2 + i\frac{7\pi}{4})\right)$$
$$= \frac{1}{8}\left(-\frac{\pi}{4}+\frac{3\pi}{4}-\frac{5\pi}{4}+\frac{7\pi}{4}\right)$$
$$= \frac{\pi}{8}$$

$$\int_0^\infty \frac{\sqrt{x}\,dx}{x^3+1} = \pi i \sum Res\left(\frac{\sqrt{z}}{z^3+1}\right)$$
$$= \pi i \sum\left(\frac{1}{3z^{3/2}}\right)_{z=z_i}$$
$$= \frac{\pi i}{3}\left(e^{-\pi i/2}+e^{-3\pi i/2}+e^{-5\pi i/2}\right)$$
$$= \frac{\pi}{3}$$