MATH 3401

TUTORIAL SHEET 6

SOLUTIONS

1. Use the fact that

$$\frac{d}{dz}\arctan z = \frac{1}{1+z^2}$$

to determine the Taylor Series expansion of $\arctan z$ in the neighbourhood of $z = 0$.

For which values of z does the series converge?

Ans

$$\begin{aligned}\frac{1}{1+z^2} &= \sum_{n=0}^{\infty}(-1)^n z^{2n} \ ; \ |z| < 1 \\ \arctan z &= \int_0^z \frac{dt}{1+t^2} \\ &= \sum_{n=0}^{\infty}(-1)^n \frac{z^{2n+1}}{2n+1} \ ; \ |z| < 1\end{aligned}$$

2. Suppose that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \ ; \ |z| < R_1$$

and that

$$g(z) = \sum_{n=0}^{\infty} b_n z^n \ ; \ |z| < R_2$$

Show that the product function $f(z)g(z)$ has the expansion

$$f(z)g(z) = \sum_{n=0}^{\infty} c_n z^n$$

where

$$c_n = \sum_{k=0}^{n} a_k b_{n-k}$$

and that the sum converges at least for $|z| < \min(R_1, R_2)$.

Ans Let $R = \min(R_1, R_2)$. Since f and g are analytic for $|z| < R$, they are regular. Therefore the product fg is also regular for $|z| < R$ and has the Taylor series expansion

$$fg(z) = \sum_{n=0}^{\infty} \frac{(fg)^{(n)}(0)}{n!} z^n$$

which converges for $|z| < R$ (at least).

By Leibniz Rule,

$$\begin{aligned} D^n(fg) &= \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \left[D^k f\right] \left[D^{n-k} g\right] \\ c_n &= \frac{1}{n!} D^n(fg)(0) \\ &= \sum_{k=0}^{n} \left[\frac{D^k f(0)}{k!}\right] \left[\frac{D^{n-k} g(0)}{(n-k)!}\right] \\ &= \sum_{k=0}^{n} a_k b_{n-k} \end{aligned}$$

3. Show that all the zeros of the polynomial

$$z^4 + 4z + 6$$

lie in the annulus $1 < |z| < 2$.

How many of the zeros lie in the left half plane?

Ans On $|z| = 1$, let $f(z) = 6$ and $g(z) = z^4 + 4z$.
Then $|f| = 6$, and $|g| \leq |z|^4 + 4|z| \leq 5 < 6$.
Therefore by Rouché's Theorem, $f + g$ has as many zeros inside the circle $|z| = 1$ as the polynomial f; namely none.

On $|z| = 2$, let $f(z) = z^4$ and $g(z) = 4z + 6$.
Then $|f| = |z|^4 = 16$, and $|g| \leq 4|z| + 6 \leq 14 < 16$.
Therefore by Rouché's Theorem, $f + g$ has as many zeros inside the circle $|z| = 2$ as the polynomial f; namely four.

Combining these results we see that all the zeros of the polynomial lie in the annulus $1 < |z| < 2$.

To determine the number of zeros in the left half plane, we consider the curve consisting of the imaginary axis and an "infinite" semicircle closed on the left.

The variation in the argument of the polynomial on the semicircular arc is 4π, so that the total variation around the contour is

$$4\pi + 2 \text{ Var } \arg(y^4 + 4iy + 6)\Big|_{y=0}^{\infty} = 4\pi$$

so that the number of zeros in the left half plane is two.

4. If $a > e$, show that the equation

$$e^z = az^n$$

has n roots inside the unit circle.

Ans Let $f = az^n$ and $g = -e^z$.
On the circle $|z| = 1$, $|f| = a|z|^n = a$, and $|g| = e^x \leq e < a$.
Therefore by Rouché's Theorem, $f + g$ has as many zeros inside the unit circle as f, namely n.