

MATH 3401
TUTORIAL SHEET 5
SOLUTIONS

1. Find the maximum of $|f(z)|$ in $|z| \leq 1$ for the functions $f(z)$ given by

- (a) $z^2 - 3z + 2$;
 (b) $z^4 + z^2 + 1$;
 (c) $\cos 3z$;
 (d) $\frac{2z + 1}{2z - 1}$.

Ans: Provided $f(z)$ is regular inside and on the unit circle, the maximum will occur on the unit circle. This applies in cases (a), (b), and (c). In case (d), the function is not regular at $z = \frac{1}{2}$ inside the circle, and $|f| \rightarrow \infty$ as $z \rightarrow \frac{1}{2}$.

Otherwise, set $z = e^{i\theta}$ and determine $|f|^2$. The maximum of $|f|^2$ will then give the maximum of $|f|$.

(a)
$$\begin{aligned} z^2 - 3z + 2 &= e^{i2\theta} - 3e^{i\theta} + 2 \\ &= (\cos(2\theta) - 3\cos(\theta) + 2) + i(\sin(2\theta) - 3\sin(\theta)) \\ |f|^2 &= \cos^2(2\theta) + 9\cos^2(\theta) + 4 + 4\cos(2\theta) - 6\cos(2\theta)\cos(\theta) - 12\cos(\theta) \\ &\quad + \sin^2(2\theta) + 9\sin^2(\theta) - 6\sin(2\theta)\sin(\theta) \\ &= 14 - 6\cos(\theta) - 12\cos(\theta) + 4\cos(2\theta) \\ &= 14 - 18\cos(\theta) + 4\cos(2\theta) \end{aligned}$$

which has a maximum value of 36 when $\cos \theta = -1$.

Therefore the maximum for $|f(z)|$ is 6 when $z = -1$.

(b)
$$\begin{aligned} z^4 + z^2 + 1 &= e^{i4\theta} + e^{i2\theta} + 1 \\ &= (\cos(4\theta) + \cos(2\theta) + 1) + i(\sin(4\theta) + \sin(2\theta)) \\ |f|^2 &= \cos^2(4\theta) + \cos^2(2\theta) + 1 + 2\cos(4\theta) + 2\cos(2\theta) + 2\cos(4\theta)\cos(2\theta) \\ &\quad + \sin^2(4\theta) + \sin^2(2\theta) + 2\sin(4\theta)\sin(2\theta) \\ &= 3 + 2\cos(4\theta) + 2\cos(2\theta) + 2\cos(2\theta) \end{aligned}$$

which has a maximum value of 9 when $\cos(2\theta) = 1$.

Therefore the maximum for $|f(z)|$ is 3 when $z = \pm 1$.

(c)
$$\begin{aligned} |\cos 3z|^2 &= \cos^2(3\cos \theta) + \sinh^2(3\sin \theta) \\ \frac{d}{d\theta}|f|^2 &= -6\cos(3\cos \theta)\sin(3\cos \theta)\sin \theta \\ &\quad + 6\sinh(3\sin \theta)\cosh(3\sin \theta)\cos \theta \\ &= 0 \text{ when } \sin \theta = 0 \text{ or } \cos \theta = 0 \end{aligned}$$

The maximum corresponds to $\cos \theta = 0$. It takes the value $\cosh 3$ when $z = \pm i$.

2. Evaluate

$$\frac{1}{2\pi i} \oint_C \frac{ze^{zt}}{(z+1)^3} dz$$

where C is any scroc enclosing $z = -1$.

Ans: This is, apart from the factor $2!$, the Cauchy integral formula for the second derivative of ze^{zt} at $z = -1$.

Therefore its value is

$$\frac{1}{2!} [2te^{zt} + t^2ze^{zt}]_{z=-1} = \frac{1}{2}(2t - t^2)e^{-t}$$

3. Let

$$f_n(z) = \frac{1}{z-n} + \frac{1}{n},$$

and let S be the disc $|z| \leq N + \frac{1}{2}$ except for the points $z = 1, 2, \dots, N$.

Use Weierstrass' M-test to show that

$$\sum_{n=1}^{\infty} f_n(z)$$

converges absolutely and uniformly on S .

(*Hint:* Consider $n \geq 2N + 1$.)

Ans: We can write

$$f_n(z) = \frac{z}{n(z-n)}$$

In S , $|z| \leq N + \frac{1}{2}$, so that if $n > 2N + 1$,

$$|z-n| \geq n - |z| > \frac{1}{2}n$$

$$\left| \frac{z}{n(z-n)} \right| < \frac{N + \frac{1}{2}}{\frac{1}{2}n^2} = \frac{2N+1}{n^2}$$

Since $\sum n^{-2}$ converges, the Weierstrass M-test shows that the series of functions converges absolutely and uniformly on S .

(This is an example of a *Mittag-Leffler expansion*.)

4. Let $F(z)$ be the limit function from question 3.

(a) Evaluate

$$\frac{1}{2\pi i} \oint_{\partial S} F(z) dz$$

where ∂S is the boundary of the region S .

Ans: Since z/n is a regular function, the Cauchy integral formula gives

$$\frac{1}{2\pi i} \oint_{\partial S} \frac{z/n}{z-n} dz = \begin{cases} 1 & n \in S \\ 0 & n \notin S \end{cases}$$

Therefore

$$\frac{1}{2\pi i} \oint_{\partial S} F(z) dz = \sum_{n=1}^{\infty} \left(\frac{1}{2\pi i} \oint_{\partial S} f_n(z) dz \right) = N$$

(b) Determine the linear approximation to $F(z)$ in the neighbourhood of $z = 0$.

Ans:

$$\begin{aligned} F(0) &= \sum_{n=1}^{\infty} f_n(0) = 0 \\ F'(0) &= \sum_{n=1}^{\infty} f'_n(0) \\ &= - \sum_{n=1}^{\infty} \frac{1}{n^2} = -\frac{\pi^2}{6} \\ F(z) &\simeq -\frac{\pi^2}{6}z \text{ near } z = 0 \end{aligned}$$