MATH 3401

Tutorial Sheet 4

Answers and Solutions

1. We have $z = e^{i\theta}$ so that $dz = ie^{i\theta} d\theta$.

(a)
$$\oint z^2\,dz = \int_0^{2\pi} e^{i2\theta} ie^{i\theta}\,d\theta = \frac{1}{3}e^{i3\theta}\Big|_0^{2\pi} = 0$$

(b)
$$\oint \frac{1}{z}\,dz = \int_0^{2\pi} \frac{ie^{i\theta}}{e^{i\theta}}\,d\theta = 2\pi i$$

(c)
$$\oint \bar{z}\,dz = \int_0^{2\pi} e^{-i\theta} ie^{i\theta}\,d\theta = 2\pi i$$

(d)
$$\oint z\bar{z}\,dz = \int_0^{2\pi} e^{i\theta}e^{-i\theta} ie^{i\theta}\,d\theta = e^{2\pi i} - 1 = 0$$

2.(c) For the segment between $1 - i$ and $1 + i$,

$$z = 1 + iy,\ -1 \le y \le 1$$
$$dz = idy\ ,\quad \bar{z} = 1 - iy$$
$$\int_{-1}^{1} (1 - iy)\,i\,dy = iy + \frac{1}{2}y^2\Big|_{-1}^{1} = 2i$$

For the segment between $1 + i$ and $-1 + i$,

$$z = x + i,\ 1 \ge x \ge -1$$
$$dz = dx\ ,\quad \bar{z} = x - i$$
$$\int_{1}^{-1} (x - i)\,dx = \frac{1}{2}x^2 - ix\Big|_{1}^{-1} = 2i$$

For the segment between $-1 + i$ and $-1 - i$,

$$z = -1 + iy,\ 1 \ge y \ge -1$$
$$dz = idy\ ,\quad \bar{z} = -1 - iy$$
$$\int_{1}^{-1} (-1 - iy)\,i\,dy = -iy + \frac{1}{2}y^2\Big|_{1}^{-1} = 2i$$

For the segment between $-1 - i$ and $1 - i$,

$$z = x - i,\ -1 \le x \le 1$$
$$dz = dx\ ,\quad \bar{z} = x + i$$
$$\int_{-1}^{1} (x + i)\,dx = \frac{1}{2}x^2 + ix\Big|_{-1}^{1} = 2i$$
$$\oint \bar{z}\,dz = 2i + 2i + 2i + 2i = 8i$$

(d) For the segment between $1-i$ and $1+i$,

$$z = 1 + iy,\ -1 \le y \le 1$$
$$dz = idy\ ,\quad z\bar{z} = 1 + y^2$$
$$\int_{-1}^{1} (1+y^2)\, i\, dy = iy + i\frac{1}{3}y^3\Big|_{-1}^{1} = \frac{8}{3}i$$

For the segment between $1+i$ and $-1+i$,

$$z = x + i,\ 1 \ge x \ge -1$$
$$dz = dx\ ,\quad z\bar{z} = x^2 + 1$$
$$\int_{1}^{-1} (x^2+1)\, dx = \frac{1}{3}x^3 + x\Big|_{1}^{-1} = -\frac{8}{3}$$

For the segment between $-1+i$ and $-1-i$,

$$z = -1 + iy,\ 1 \ge y \ge -1$$
$$dz = idy\ ,\quad z\bar{z} = 1 + y^2$$
$$\int_{1}^{-1} (1+y^2)\, i\, dy = iy + i\frac{1}{3}y^3\Big|_{1}^{-1} = -\frac{8}{3}i$$

For the segment between $-1-i$ and $1-i$,

$$z = x - i,\ -1 \le x \le 1$$
$$dz = dx\ ,\quad z\bar{z} = x^2 + 1$$
$$\int_{-1}^{1} (x^2+1)\, dx = \frac{1}{3}x^3 + x\Big|_{-1}^{1} = \frac{8}{3}$$
$$\oint z\bar{z}\, dz = \frac{8}{3}i - \frac{8}{3} - \frac{8}{3}i + \frac{8}{3} = 0$$

3. Since the integrand is entire, $\oint e^{-z^2}\,dz = 0$.
On the first part of the contour

$$z = x\ ;\ dz = dx$$
$$\int e^{-z^2}\,dz = \int_0^R e^{-x^2}\,dx$$

On the second part of the contour

$$z = R + iy\ ;\ dz = idy$$
$$\int e^{-z^2}\,dz = \int_0^a \exp(-R^2 + y^2 - 2iRy)(i\,dy)$$
$$\left|\exp(-R^2 + y^2 - 2iRy)\right| = \exp(y^2 - R^2) \le \exp(a^2 - R^2)$$
$$\left|\int_0^a \exp(-R^2 + y^2 - 2iRy)(i\,dy)\right| \le a e^{a^2 - R^2} \downarrow 0 \text{ as } R \to \infty$$

On the third part of the contour

$$z = x + ia\ ;\ dz = dx$$
$$\int e^{-z^2}\,dz = \int_R^0 \exp(-x^2 + a^2 - 2iax)\,dx$$
$$= -e^{a^2} \int_0^R e^{-x^2} \cos(2ax)\,dx + i e^{a^2} \int_0^R e^{-x^2} \sin(2ax)\,dx$$

On the fourth part of the contour

$$z = iy\ ;\ dz = idy$$
$$\int e^{-z^2}\,dz = \int_a^0 e^{y^2}\,(i\,dy) = -i \int_0^a e^{y^2}\,dy$$

Combining these results we have

$$\left(\int_0^R e^{-x^2}\,dx - e^{a^2} \int_0^R e^{-x^2} \cos(2ax)\,dx\right)$$
$$+i\left(e^{a^2} \int_0^R e^{-x^2} \sin(2ax)\,dx - \int_0^a e^{y^2}\,dy\right)$$
$$+O\left(e^{-R^2}\right) = 0$$

Taking the limit as $R \to \infty$, and equating real and imaginary parts we obtain

$$\int_0^\infty e^{-x^2} \cos(2ax)\,dx = e^{-a^2} \int_0^\infty e^{-x^2}\,dx = e^{-a^2} \frac{\sqrt{\pi}}{2}$$

and

$$\int_0^\infty e^{-x^2} \sin(2ax)\,dx = e^{-a^2} \int_0^a e^{y^2}\,dy$$

4.(a) Inside $|z| = 1$, the integrand is regular, therefore

$$\oint_C \frac{e^z}{z^2 - 9}\, dz = 0 \ .$$

(b) Writing $f(z) = e^z/(z+3)$, we have

$$\oint_C \frac{e^z}{z^2 - 9}\, dz = \oint_C \frac{f(z)}{z - 3}\, dz = 2\pi i f(3) = \frac{\pi i}{3} e^3$$

(c) Writing $f(z) = e^z/(z-3)$, we have

$$\oint_C \frac{e^z}{z^2 - 9}\, dz = \oint_C \frac{f(z)}{z + 3}\, dz = 2\pi i f(-3) = -\frac{\pi i}{3} e^{-3}$$

5. The term

$$\frac{w(\zeta) - w(z)}{\zeta - z}$$

is a polynomial of degree $n-1$ in z.

Writing this as $\sum a_r(\zeta) z^r$, we have

$$\begin{aligned} P(z) &= \frac{1}{2\pi i} \oint_c \frac{f(\zeta)}{w(\zeta)} \sum_{r=0}^{n-1} a_r(\zeta) z^r \, d\zeta \\ &= \sum_{r=0}^{n-1} z^r \left(\frac{1}{2\pi i} \oint_c \frac{f(\zeta)}{w(\zeta)} a_r(\zeta)\, d\zeta \right) \\ &= \sum_{r=0}^{n-1} A_r z^r \end{aligned}$$

which shows that $P(z)$ is a polynomial of degree $n-1$ in z.

Since $w(z_i) = 0$,

$$\begin{aligned} P(z_i) &= \frac{1}{2\pi i} \oint_c \frac{f(\zeta)}{w(\zeta)} \frac{w(\zeta) - w(z_i)}{\zeta - z_i}\, d\zeta \\ &= \frac{1}{2\pi i} \oint_c \frac{f(\zeta)}{\zeta - z_i}\, d\zeta \\ &= f(z_i) \end{aligned}$$