MATH 3401

Tutorial Sheet 3

Answers and Solutions

1.(i) $u(x,y) = x^2 - y^2 + 2x$, $v(x,y) = 2xy + 2y$
(ii) $u(x,y) = x/(x^2+y^2)$, $v(x,y) = -y/(x^2+y^2)$
(iii) $u(x,y) = x^2 + y^2 - 1$, $v(x,y) = -2y$.

2.

(i)
$$u_x = 2x + 2 \quad v_x = 2y$$
$$f'(z) = 2x + 2 + i2y = 2z + 2$$

(ii)
$$u_x = \frac{1}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2}$$
$$v_x = \frac{2xy}{(x^2+y^2)^2}$$
$$f'(z) = -\frac{x^2 - y^2 - i2xy}{(x^2+y^2)^2} = -\frac{(\bar{z})^2}{(z\bar{z})^2} = -1/z^2$$

3. $u_x = 2x$, $v_y = -2$, $u_x = v_y$ when $x = -1$.
$u_y = 2y$, $v_x = 0$, $u_y = -v_x$ when $y = 0$.
Therefore this function is not differentiable if $z \neq -1$.

4.(a) $u_x = 2(1-y)$, $u_{xx} = 0$, $u_y = -2x$, $u_{yy} = 0$.
$u_{xx} + u_{yy} = 0$ for all x, y .
$v_y = u_x = 2 - 2y$, $v = 2y - y^2 + c(x)$, $v_x = c'(x) = -u_y = 2x$ $c(x) = x^2(+b)$
$v = x^2 - y^2 + 2y$.

(b) $u_x = 2 + 6xy$, $u_{xx} = 6y$, $u_y = -3y^2 + 3x^2$, $u_{yy} = -6y$.
$u_{xx} + u_{yy} = 0$ for all x, y .
$v_y = u_x = 2 + 6xy$, $v = 2y + 3xy^2 + c(x)$, $v_x = 3y^2 + c'(x) = -u_y = 3y^2 - 3x^2$.
$c'(x) = -3x^2$, $c(x) = -x^3$, $v = 2y + 3xy^2 - x^3$.

(c) $u_x = \cosh x \sin y$, $u_{xx} = \sinh x \sin y$, $u_y = \sinh x \cos y$, $u_{yy} = -\sinh x \sin y$.
$u_{xx} + u_{yy} = 0$ for all x, y .
$v_x = -u_y = -\sinh x \cos y$, $v = -\cosh x \cos y + c(y)$, $v_y = \cosh x \sin y + c'(y) = u_x = \cosh x \sin y$
$c'(y) = 0$, $c = 0$, $v = -\cosh x \cos y$

(d)
$$u_x = -\frac{2xy}{(x^2+y^2)^2} \quad u_{xx} = -\frac{2y}{(x^2+y^2)^2} + \frac{8x^2y}{(x^2+y^2)^3} = \frac{6x^2y - 2y^3}{(x^2+y^2)^3}$$
$$u_y = \frac{1}{x^2+y^2} - \frac{2y^2}{(x^2+y^2)^2} = \frac{x^2-y^2}{(x^2+y^2)^2}$$
$$u_{yy} = -\frac{2y}{(x^2+y^2)^2} - \frac{4x^2y - 4y^3}{(x^2+y^2)^3} = \frac{2y^3 - 6x^2y}{(x^2+y^2)^3}$$
$$u_{xx} + u_{yy} = 0 \quad \text{for all} \quad (x,y) \neq (0,0)$$

$$v_y = u_x = -\frac{2xy}{(x^2+y^2)^2} \quad v = \frac{x}{x^2+y^2}$$