

MATH 3401
TUTORIAL SHEET 2
SOLUTIONS

1. A sequence $\{w_n\}$ is defined for all $z \in \mathbb{C}$ by

$$w_0 = \pi z ,$$

$$w_n = \left(1 - \frac{z^2}{n^2}\right) w_{n-1} .$$

Calculate w_5 for the cases

(a) $z = \frac{1}{2}$, (b) $z = \frac{1}{6}$, (c) $z = \frac{i}{2}$.

(a) $\{1.5708, 1.1781, 1.1045, 1.0738, 1.0570, 1.0464\}$

(b) $\{.5236, .5091, .5075, .5066, .5060\}$

(c) $\{1.5708i, 1.9635i, 2.0802i, 2.1442i, 2.1777i, 2.1944i\}$

Given that these sequences converge, what do you imagine are the limits in cases (a) and (b)?

This infinite product converges to $\sin(\pi z)$.

The limits for (a) and (b) are therefore 1 and .5 respectively.

2. A sequence $\{w_n\}$ is defined for each $z \in \mathbb{C}$ by

$$w_1 = 2 ,$$

$$w_{n+1} = 2 + \frac{z}{w_n} .$$

Calculate w_5 when $z = -1 + 2i$.

$\{\frac{3}{2} + i, \frac{28}{13} + \frac{16}{13}i, \frac{41}{20} + \frac{9}{10}i, \frac{782}{401} + \frac{400}{401}i\}$
(This sequence converges to $2 + i$)

3. Find all values of z such that

- (a) $\exp z = 1 + i\sqrt{3}$
(b) $\exp z = -i$

- (a) $e^x \cos y = 1$
 $e^x \sin y = \sqrt{3}$
 $e^{2x}(\cos^2 y + \sin^2 y) = 1 + 3 = 4$
 $e^x = 2 ; x = \log 2 = 0.693147181\dots$
 $\cos y = \frac{1}{2} \quad \sin y = \frac{\sqrt{3}}{2}$
 $y = \frac{\pi}{3} + 2n\pi$
 $z = \log 2 + (2n + 1/3)\pi i$

$$\begin{aligned}
\text{(b)} \quad e^x \cos y &= 0 \\
e^x \sin y &= -1 \\
e^x &= 1 ; x = 0 \\
\cos y = 0 \quad \sin y &= -1 \\
y &= -\frac{\pi}{2} + 2n\pi \\
z &= (2n - 0.5)\pi i
\end{aligned}$$

4. Find all roots of the equations

$$\text{(a)} \quad \sin z = 2$$

$$\text{(b)} \quad \cosh z = \frac{1}{2}$$

$$\text{(a)} \quad \sin z = \sin x \cosh y + i \cos x \sinh y = 2$$

$$\sin x \cosh y = 2$$

$$\cos x \sinh y = 0 ; \cos x = 0 \text{ or } \sinh y = 0$$

If $\sinh y = 0 ; \cosh y = 1 ; \sin x = 2 ;$ impossible

$$\cos x = 0 ; \sin x = \pm 1 ; \cosh y = \pm 2$$

Since $\cosh y > 0 ; \cosh y = 2 ; y = \pm \operatorname{arccosh} 2 = \pm \log(2 + \sqrt{3})$

$$\sin x = 1 ; x = (2n + \frac{1}{2})\pi$$

$$z = (2n + \frac{1}{2})\pi \pm i \log(2 + \sqrt{3})$$

$$\text{(b)} \quad \cosh z = \cosh x \cos y + i \sinh x \sin y = \frac{1}{2}$$

$$\sinh x \sin y = 0 ; x = 0 \text{ or } \sin y = 0$$

If $\sin y = 0 ; \cos y = \pm 1 ; \cosh x = \pm 0.5$ impossible

therefore $\cos y = 0.5 ; y = \pm \frac{\pi}{3} + 2n\pi$

$$z = \left(2n \pm \frac{1}{3}\right) \pi i$$

5. Evaluate i^i .

$$i^i = \exp(i \log i) = \exp(i(\log(1) + (2n + 0.5)\pi i)) = \exp(-(2n + 0.5)\pi)$$