MATH 3401

TUTORIAL SHEET 10

SOLUTIONS

1. The transform which sends $0 \to i$, $1 \to 1$ and $-1 \to 0$ is given by

$$\frac{(w-i)(0-1)}{(w-1)(0-i)} = \frac{(z-0)(-1-1)}{(z-1)(-1-0)}$$
$$-i(w-i)(z-1) = 2(w-1)z$$
$$(2+i)wz - iw = z+1$$
$$w = \frac{z+1}{(2+i)z-i}$$

2. If w maps the upper half plane onto the unit circle, and maps $z = i$ onto $w = 0$, then

$$w = e^{i\theta}\frac{z-i}{z+i}$$

If in addition $z = \infty$ maps onto $w = -1$, then

$$-1 = e^{i\theta}$$
$$w = -\frac{z-i}{z+i}$$

3. If w maps the unit circle onto itself, then it has the form

$$w = e^{i\theta}\frac{z-c}{1-c^*z}$$

If $z = 0$ maps onto $w = \frac{1}{2}$,

$$\frac{1}{2} = e^{i\theta}(-c)$$

and if $z = 1$ maps onto $w = 1$.

$$1 = e^{i\theta}\frac{1-c}{1-c^*}$$
$$-2c = \frac{1-c}{1-c^*}$$
$$-2c + 2cc^* = 1-c$$
$$c = 2cc^* - 1 = -\frac{1}{2}$$
$$w = \frac{2z+1}{2+z}$$

4. The common diameter of the two circles intersects $\mathcal{C}_1$ at $z = 1$ and $z = -1$, and $\mathcal{C}_2$ at $z = \frac{7}{2}$ and $z = -\frac{3}{2}$.

Therefore a possible transform will map $1 \to 1$, $-1 \to -1$, $\frac{7}{2} \to \rho$ and $-\frac{3}{2} \to -\rho$, where $\rho > 1$.

This gives

$$\left(\frac{w-1}{w+1}\right)\left(\frac{\rho+1}{\rho-1}\right) = \left(\frac{z-1}{z+1}\right)\left(\frac{\frac{7}{2}+1}{\frac{7}{2}-1}\right)$$

$$\left(\frac{-\rho-1}{-\rho+1}\right)\left(\frac{\rho+1}{\rho-1}\right) = \left(\frac{-\frac{3}{2}-1}{-\frac{3}{2}+1}\right)\left(\frac{\frac{7}{2}+1}{\frac{7}{2}-1}\right)$$

$$\left(\frac{\rho+1}{\rho-1}\right)^2 = 9$$

$$\left(\frac{\rho+1}{\rho-1}\right) = 3 \; ; \; \rho = 2$$

$$3\frac{w-1}{w+1} = \frac{9}{5}\frac{z-1}{z+1}$$

$$5(w-1)(z+1) = 3(w+1)(z-1)$$

$$2wz + 8w = 8z + 2$$

$$w = \frac{4z+1}{z+4}$$