MATH3401

TUTORIAL SHEET 1

SOLUTIONS

1. Simplify each of the following expressions into the form $a + ib$, where a and b are real.

(i)
$$\frac{1 + i\tan\alpha}{1 - i\tan\alpha}\ ,$$
$$\cos(2\alpha) + i\sin(2\alpha)$$

(ii)
$$\frac{(1+2i)^3 - (1-i)^3}{(3+2i)^3 - (2+i)^2}\ ,$$
$$\frac{27 + 99i}{520}$$

(iii)
$$\frac{(1-i)^5 - 1}{(1+i)^5 + 1}\ ,$$
$$\frac{-1 - 32i}{25}$$

(iv)
$$\frac{(1+i)^9}{(1-i)^7}\ .$$
$$2$$

2. Find the modulus and argument of

(a)
$$5 + 12i\ ,$$
$$|5 + 12i| = 13\ ;\ \arg(5 + 12i) = 1.176^c \text{ or } 67.38^o$$

(b)
$$\frac{-1 + i}{\sqrt{2}}\ .$$
$$|z| = 1\ ;\ \arg z = \frac{3\pi}{4}^c \text{ or } 135^o$$

Hence find all solutions of

(c)
$$z^2 = 5 + 12i\ ,$$
$$z = \pm(3 + 2i)$$

(d)
$$z^3 = \frac{-1 + i}{\sqrt{2}}\ .$$
$$z = \frac{1 + i}{\sqrt{2}}\omega^k\ ;\ k = 0, 1, 2$$
$$\omega = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

3. Let $\omega = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$.
(*Note that* $\omega^3 = 1$ *and* $1 + \omega + \omega^2 = 0$)
Calculate

(i)
$$(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega) \ ,$$
$$a^2 + b^2 + c^2 - ab - bc - ca$$

(ii)
$$(a + b)(a + b\omega)(a + b\omega^2) \ ,$$
$$a^3 + b^3$$

(iii)
$$(a + b\omega + c\omega^2)^3 + (a + b\omega^2 + c\omega)^3 \ ,$$
$$2(a^3 + b^3 + c^3) - 3(a^2b + b^2c + c^2a + a^2c + b^2a + c^2b) + 12abc$$

(iv)
$$(a\omega^2 + b\omega)(b\omega^2 + a\omega) \ .$$
$$a^2 - ab + b^2$$

4. Let ω be an nth root of unity; that is, $\omega^n = 1$.
Evaluate

(i)
$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} \ ,$$
$$\text{If } \omega = 1 \ , \ \Sigma = n$$
$$\text{If } \omega \neq 1 \ , \ (1 - \omega)\Sigma = 1 - \omega^n = 0$$
$$\Sigma = 0$$

(ii)
$$1 + 2\omega + 3\omega^2 + \cdots + n\omega^{n-1} \ ,$$
$$\text{If } \omega = 1 \ , \ \Sigma = \frac{n(n+1)}{2}$$
$$\text{If } \omega \neq 1 \ , \ (1 - \omega)\Sigma = 1 + \omega + \cdots + \omega^{n-1} - n\omega^n$$
$$\Sigma = \frac{-n}{1 - \omega}$$

(iii)
$$1 + 4\omega + 9\omega^2 + \cdots + n^2\omega^{n-1} \ .$$
$$\text{If } \omega = 1 \ , \ \Sigma = \frac{n(n+1)(2n+1)}{6}$$
$$\text{If } \omega \neq 1 \ , \ (1 - \omega)\Sigma = 1 + 3\omega + \cdots + (2n-1)\omega^{n-1} - n^2\omega^n$$
$$= 2(1 + 2\omega + \cdots + n\omega^{n-1}) - (1 + \omega + \cdots + \omega^{n-1}) - n^2$$
$$\Sigma = -\frac{n^2}{1 - \omega} - \frac{2n}{(1 - \omega)^2}$$

5. Sketch in the complex plane the regions defined by the following expressions;

(i) $$|z - i| < 2$$

The interior of the circle, centre $z = i$ and radius 2

(ii) $$\mathrm{Re}(z - 1 - i) > 2$$

The right half plane $x > 3$

(iii) $$\left|\frac{z+1}{z-1}\right| < 2$$

The exterior of the circle, centre $z = \dfrac{5}{3}$, radius $\dfrac{4}{3}$

(iv) $$\mathrm{Im}\left(\frac{z-i}{1+i}\right) < 0$$

The half plane $y < x + 1$

(v) $$\mathrm{Im}\left(\frac{1}{z}\right) < 0$$

The half plane $y > 0$

(vi) $$\left\{\mathrm{Re}\left(\frac{z-a}{b}\right) = 0\right\} \bigcup \left\{\mathrm{Im}\left(\frac{z-a}{b}\right) = 0\right\}$$

A pair of perpendicularlines passing through $z = a$, one in the direction of b

6. Sketch in the $u - v$ plane the curves into which the lines $x = \frac{1}{2}, 1, \frac{3}{2}, 2$ are mapped by the function $w = z^2$.

If $w = z^2$, then $u = x^2 - y^2$ and $v = 2xy$.
When $x = \frac{1}{2}$, $v = y$ and $u = \frac{1}{4} - y^2 = \frac{1}{4} - v^2$.
When $x = 1$, $u = 1 - \frac{1}{4}v^2$.
When $x = \frac{3}{2}$, $u = \frac{9}{4} - \frac{1}{9}v^2$.
When $x = 2$, $u = 4 - \frac{1}{16}v^2$.
All these curves are parabolas.

What happens to the interior of the square whose vertices are $\{1, 2, 2 + i, 1 + i\}$ under this mapping?

What are the angles at the corners of this new figure?

The angles at the corners of this new figure are all right angles.