Residues

If z_0 is an isolated pole or essential singular point of a regular function f, then we can expand f in a Laurent expansion about z_0.

The coefficients a_k in the expansion are given by

$$a_k = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{k+1}}\, dz$$

In particular

$$a_{-1} = \frac{1}{2\pi i} \oint f(z)\, dz$$

a_{-1} is called the *Residue of the function* $f(z)$ *at* z_0.

Alternatively

$$\oint f(z)\, dz = 2\pi i a_{-1}$$

If $f(z)$ is single valued inside and on a scroc C, and has poles/esps at z_0, z_1,... z_n inside C, then

$$\oint_C f(z)\, dz = \sum_{k=0}^{n} \oint_{C_k} f(z)\, dz$$

where C_k is a scroc $|z - z_k| = \epsilon$ enclosing z_k and no other singular points.

Therefore

$$\oint_C f(z)\, dz = 2\pi i \sum_{k=0}^{n} \left(Res\, f(z)|_{z_k} \right)$$

This result is known as *The Residue Theorem*.

Evaluating Residues

1. Simple poles.

Method 1.

If

$$f(z) = \frac{a_{-1}}{z - z_0} + g(z)$$

where $g(z)$ is regular at z_0, then

$$(z - z_0)f(z) = a_{-1} + (z - z_0)g(z)$$
$$a_{-1} = \lim_{z \to z_0} (z - z_0)f(z)$$

e.g.

$$f(z) = \frac{1}{z(z^2 - 1)}$$

has simple poles at -1, 0, and 1 .

The residues are

$$\text{at } z = -1 \;:\; \lim_{z \to -1} \frac{1}{z(z-1)} = \frac{1}{2}$$
$$\text{at } z = 0 \;:\; \lim_{z \to 0} \frac{1}{(z^2-1)} = -1$$
$$\text{at } z = 1 \;:\; \lim_{z \to 1} \frac{1}{z(z+1)} = \frac{1}{2}$$

Method 2.

If $f(z) = p(z)/q(z)$, where p and q are regular with $p(z_0) \neq 0$ and $q(z_0) = 0$, then

$$\begin{aligned}\lim_{z \to z_0} (z - z_0) \frac{p(z)}{q(z)} &= \lim_{z \to z_0} p(z) \frac{z - z_0}{q(z) - q(z_0)} \\ &= \frac{p(z_0)}{q'(z_0)}\end{aligned}$$

e.g.

$f(z) = \cot z = \cos z / \sin z$ has poles at the zeros of $\sin z$, namely at $z = n\pi$, $n = 0, \pm 1, \pm 2, \dots$.

Since $(\sin z)' = \cos z$, the residues at these poles is

$$\frac{\cos z_0}{\cos z_0} = 1$$

Method 3.

If $f(z)$ has a simple pole at z_0 with residue a_{-1}, and $g(z)$ is regular at z_0, then $f(z)g(z)$ has a simple pole at z_0 with residue $a_{-1}g(z_0)$.

e.g. $f(z) = \pi \cot(\pi z)$ has simple poles at $z = n$ with residues 1. Therefore

$$\frac{\pi \cot(\pi z)}{z^2 + 1}$$

has poles at $z = n$ with residues

$$\frac{1}{n^2 + 1}$$

It also has poles at $z = \pm i$ with residues (using method 2)

at $z = i$:

$$\frac{\pi \cot(\pi i)}{2i} = \frac{\pi \cosh(\pi)}{(i \sinh(\pi))(2i)} = -\frac{\pi}{2} \coth \pi$$

at $z = -i$:

$$\frac{\pi \cot(-\pi i)}{-2i} = \frac{\pi \cosh(\pi)}{(-i \sinh(\pi))(-2i)} = -\frac{\pi}{2} \coth \pi$$

Second order poles.

If $f(z)$ has a pole of order 2 at z_0, then $g(z) = (z - z_0)^2 f(z)$ has a removable singularity there.

$$\begin{aligned}\frac{1}{2\pi i} \oint f(z)\, dz &= \frac{1}{2\pi i} \oint \frac{g(z)}{(z - z_0)^2}\, dz \\ &= g'(z_0) \\ &\quad \text{(using the Cauchy Integral Formula)} \\ &= \lim_{z \to z_0} \frac{d}{dz} \left((z - z_0)^2 f(z) \right)\end{aligned}$$

e.g. $f(z) = 1/(z^2 + 1)^2$ has poles of order 2 at $z = \pm i$.

The residues are:
at $z = i$:

$$\lim_{z\to i} \frac{d}{dz}\left(\frac{1}{(z+i)^2}\right) = \lim_{z\to i} \frac{-2}{(z+i)^3} = \frac{-2}{(2i)^3} = -\frac{i}{4}$$

at $z = -i$:

$$\lim_{z\to -i} \frac{d}{dz}\left(\frac{1}{(z-i)^2}\right) = \lim_{z\to i} \frac{-2}{(z-i)^3} = \frac{-2}{(-2i)^3} = \frac{i}{4}$$