Polynomials and Rational functions

Polynomials.

Consider the polynomial

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$$

where $a_n \neq 0$.

The **degree** of $p(z)$, denoted $\partial(p)$, is n.

For $z \neq 0$, we can write $p(z)$ in the form

$$\begin{aligned} p(z) &= a_n z^n \left(1 + \frac{a_{n-1}}{a_n z} + \cdots + \frac{a_1}{a_n z^{n-1}} + \frac{a_0}{a_n z^n}\right) \\ &= a_n z^n \left(1 + \sum_{i=1}^{n} \frac{a_{n-i}}{a_n z^i}\right) \end{aligned}$$

Now

$$\left|\frac{a_{n-i}}{a_n z^i}\right| < \frac{1}{2n} \quad \text{if} \quad |z| > \left(\frac{2n|a_{n-i}|}{|a_n|}\right)^{1/i} \; (= r_i)$$

so that if $|z| > r = \max r_i$,

$$\left|\sum_{i=1}^{n} \frac{a_{n-i}}{a_n z^i}\right| \leq \sum_{i=1}^{n} \left|\frac{a_{n-i}}{a_n z^i}\right| < \sum_{i=1}^{n} \frac{1}{2n} = \frac{1}{2} \; .$$

Therefore, for $|z| > r$,

$$\frac{1}{2} < \left|1 + \sum_{i=1}^{n} \frac{a_{n-i}}{a_n z^i}\right| < \frac{3}{2}$$

and

$$\frac{1}{2}|a_n||z|^n < |p(z)| < \frac{3}{2}|a_n||z|^n \; .$$

Note that this shows that all the zeros of p lie inside the circle $|z| = r$ (at least).

Order notation.

We say that $f(z)$ is of the order of $g(z)$ as $|z| \to \infty$, written $f(z) = O(g(z))$, if, for some $r > 0$, $|f(z)| \leq c|g(z)|$ for all $|z| > r$.

Therefore, if p is a polynomial of degree n, $p(z) = O(z^n)$ as $|z| \to \infty$.

Rational functions.

If $p(z)$ and $q(z)$ are polynomials, then

$$f(z) = \frac{p(z)}{q(z)}$$

is a **rational** function.

The function is defined except at the zeros of $q(z)$.

By the quotient rule,

$$f'(z) = \frac{q(z)p'(z) - p(z)q'(z)}{q^2(z)} \; ,$$

so that f is regular except at the zeros of q.

The function has poles at the zeros of q.

If

$$q(z) = b_m z_m + b_{m-1} z_{m-1} + \dots b_1 z + b_0 \ ,$$

then there is a number $r_q > 0$ such that

$$\frac{1}{2}|b_m||z|^m < |q(z)| < \frac{3}{2}|b_m||z|^m$$

for all $|z| > r_q$.

Inverting this we have

$$\frac{2}{3|b_m|}|z|^{-m} < \frac{1}{|q(z|} < \frac{2}{|b_m|}|z|^{-m}$$

for all $|z| > r_q$.

Since q has no zeros for $|z| > r_q$, f is regular for $|z| > r_q$.

Furthermore, if $|z| > r$ also,

$$\frac{1}{3}\frac{|a_n|}{|b_m|}|z|^{n-m} < \frac{|p(z)|}{|q(z)|} = |f(z)| < 3\frac{|a_n|}{|b_m|}|z|^{n-m}$$

and $f(z) = O(z^{n-m})$ as $|z| \to \infty$.

Integrals and residues.

Suppose that $m = \partial(q) \geq \partial(p) + 2 = n + 2$.

Consider

$$\oint_{\mathcal{C}} f(z)\, dz = \oint_{\mathcal{C}} \frac{p(z)}{q(z)}\, dz \ ,$$

where we take as our contour $\mathcal{C}$ the circle $|z| = R$, where $R > 1$ is large enough to enclose all the zeros of q.

Then on this contour

$$\left|\frac{p(z)}{q(z)}\right| < cR^{n-m} \leq cR^{-2}$$

for some constant c, so that

$$\left|\oint_{\mathcal{C}} \frac{p(z)}{q(z)}\, dz\right| \leq \frac{c}{R^2}\, 2\pi R = \frac{2\pi c}{R} \ ,$$

which can be made as small as we like by taking R sufficiently large.

By the Residue Theorem, the value of this integral is $2\pi i$ times the sum of the residues of f inside the contour; i.e. $2\pi i$ times the sum of all the residues of f.

This value is independent of R, so that we have

$$\oint_{\mathcal{C}} f(z)\, dz = 0 \ ,$$

and the sum of the residues of f is also zero.

Note that this result only holds if $\partial(q) \geq \partial(p) + 2$.

Improper Real Integrals

The integral

$$\int_a^\infty f(x)\,dx$$

is defined as

$$\lim_{X\to\infty}\int_a^X f(x)\,dx$$

provided that the limit exists.

Suppose that

$$f(x) = \frac{p(x)}{q(x)} = \frac{x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0}{x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0} .$$

There is a value r such that if $x > r$,

$$\frac{1}{3}x^{n-m} < f(x) < 3x^{n-m} .$$

We can split our integral into

$$\int_a^X f(x)\,dx = \int_a^r f(x)\,dx + \int_r^X f(x)\,dx .$$

The first term on the righthand side is a constant, so that the convergence or divergence of the integral depends on

$$\lim_{X\to\infty}\int_r^X f(x)\,dx .$$

Since the integrand is positive for $x > r$, the integral is a monotonic increasing function of X, and hence converges if and only if it is bounded.

If $(n-m) < -1$,

$$\begin{aligned}\int_r^X f(x)\,dx &\le 3\int_r^X x^{n-m}\,dx \\ &= \frac{3}{n-m+1}\quad x^{n-m+1}\Big|_r^X \\ &= \frac{3}{m-n-1}\left(\frac{1}{r^{m-n-1}} - \frac{1}{X^{m-n-1}}\right) \\ &< \frac{3}{m-n-1}\frac{1}{r^{m-n-1}}\end{aligned}$$

so that the integral converges.

On the other hand, if $(n-m) \ge -1$,

$$\begin{aligned}\int_r^X f(x)\,dx &\ge \frac{1}{3}\int_r^X x^{n-m}\,dx \\ &\ge \frac{1}{3}\int_r^X \frac{dx}{x} \\ &= \frac{1}{3}\log\left(\frac{X}{r}\right)\end{aligned}$$

which is unbounded as $X \to \infty$, so that the integral diverges.

Therefore, for the convergence of the improper integral of a rational function, we require $\partial(q) \ge \partial(p) + 2$.

Algebraic multiplicity

If $p(z)$ is a polynomial of degree n in z, then, by the fundamantal theorem of algebra, the polynomial equation

$$p(z) = a$$

has precisely n solutions for every complex number a.

The polynomial therefore represents an $n - 1$ mapping of $\mathbb{C}$ onto itself. In particular, the mapping is $1 - 1$ if and only if $\partial(p) = 1$.

For rational functions it is more convenient to consider the extended complex plane - Riemann sphere.

If

$$f(z) = \frac{p(z)}{q(z)}$$

where p and q are polynomials, $\partial(p) = n$, $\partial(q) = m$, then the equation

$$f(z) = a$$

is equivalent to the polynomial equation

$$p(z) - aq(z) = 0 \ .$$

Setting $M = \max(n, m)$, the degree of this equation is M in general, and if $a \in \mathbb{C}$, there are M solutions in $\mathbb{C}$.

The exceptional cases require the use of the point at infinity.

1. If $n = m = M$, then for $a = a_n/b_n$, $p(z) - aq(z)$ is a polynomial of degree less than M.

However, in this case

$$\lim_{z \to \infty} \frac{p(z)}{q(z)} = \frac{a_n}{b_n} = a$$

so that $z = \infty$ supplies the missing solution(s).

2. If $n < m = M$, then for $a = 0$, the polynomial equation $p(z) = 0$ has only n roots. The remaining $m - n$ solutions of $f(z) = 0$ are supplied by $z = \infty$, and we say that f has a zero of order $m - n$ at infinity.

3. If $m < n = M$, then the m zeros of q in $\mathbb{C}$ map onto the point at infinity. The remaining $n - m$ solutions of $f(z) = \infty$ are again supplied by $z = \infty$. In this case we say that f has a pole of order $n - m$ at infinity.

A rational function therefore represents an $M - 1$ mapping of the extended complex plane onto itself.

In particular the **linear fractional transformation**

$$w = \frac{az + b}{cz + d} \quad ; \quad ad - bc \neq 0 \ ,$$

is a $1 - 1$ mapping if the extended complex plane onto itself.