

A CLASS OF POLYNOMIALS IN \mathbb{C} .

For $Q = 0, 1, \dots, N-1$; $L = 1, \dots, N$, consider

$$\begin{aligned} P_{(L)}^{(Q,N)}(z) &= \sum_{j=0}^{N-1} \omega^{jQ} \sum_{k=0}^{N-1} (z^{N-k-1} \omega^{-jk})^L \\ &= \sum_{k=0}^{N-1} (z^L)^{N-k-1} \sum_{j=0}^{N-1} \omega^{j(Q-kL)} \end{aligned}$$

where $\omega = e^{2\pi i/N}$.

$$\sum_{j=0}^{N-1} \omega^{j(Q-kL)} = \frac{1 - \omega^{N(Q-kL)}}{1 - \omega^{Q-kL}} = 0$$

unless $\omega^{Q-kL} = 1$.

Hence the only non-trivial terms in the expansion occur when $Q \equiv kL \pmod{N}$.

Let d be the greatest common divisor of L and N .

If d does not divide Q , the congruence has no solutions, and $P \equiv 0$.

Otherwise, the congruence has d solutions $k^* + r(N/d)$ where $0 \leq k^* < N/d$ and $r = 0, 1, \dots, d-1$.

In this case (setting $\zeta = z^L$)

$$\begin{aligned} P &= N \sum_{r=0}^{d-1} \zeta^{N-k^*-1-r(N/d)} \\ &= N \zeta^{N-k^*-1} \sum_{r=0}^{d-1} (\zeta^{N/d})^{-r} \\ &= N \zeta^{N-k^*-1} \frac{1 - \zeta^{-N}}{1 - \zeta^{-N/d}} \\ &= N \zeta^{(N/d)-k^*-1} \frac{\zeta^N - 1}{\zeta^{N/d} - 1} \end{aligned}$$

which has, in terms of ζ , $(N/d) - k^* - 1$ zeros when $\zeta = 0$ and $N - (N/d)$ zeros **on the unit circle**.

Since $\zeta = z^L$, the polynomial in z has $(L/d)N - k^*L - L$ zeros at the origin and $LN - (L/d)N$ zeros on the unit circle.

Apart from the factor z^{-k^*L-L} , the polynomial has the expansion $t + t^2 + \dots + t^d$, where $t = z^{NL/d}$. This shows that $z = 1$ is never a root of the polynomial. For $z = -1$, it could be a root provided d is even and NL/d is odd. However L/d is an integer and N is a multiple of d , so these conditions are incompatible. Hence the only real roots are at $z = 0$.

In view of the earlier congruence, $L + k^*L \equiv L + Q \pmod{N}$.

If N is prime, and $L < N$, $d = 1$, the congruence has a unique solution (mod(N)) for every choice of Q , and $P = Nz^{NL-k^*L-L}$.

When $L = N$, prime or composite, the congruence is only solvable when $Q = 0$, so that for $Q \geq 1$, $P \equiv 0$, while when $Q = 0$, $k^* = 0$, and $P = N(1 + z^N + \cdots + z^{N^2-N})$.

For general N , if $L = 1$, then $d = 1$ and $k^* = Q$, so that

$$P_{(1)}^{(Q,N)}(z) = Nz^{N-Q-1} .$$

Similarly, for general N , if $L = N - 1$, then $d = 1$ and $k^* = 0$ when $Q = 0$ and $k^* = N - Q$ otherwise. Therefore

$$P_{(N-1)}^{(0,N)}(z) = Nz^{(N-1)^2}$$

$$P_{(N-1)}^{(Q,N)}(z) = Nz^{(N-1)(Q-1)}$$

For particular cases we have:

$N = 3$;

	$Q = 0$	$Q = 1$	$Q = 2$
$L = 1$	$3z^2$	$3z$	3
$L = 2$	$3z^4$	3	$3z^2$
$L = 3$	$3 + 3z^3 + 3z^6$	0	0

$N = 4$;

	$Q = 0$	$Q = 1$	$Q = 2$	$Q = 3$
$L = 1$	$4z^3$	$4z^2$	$4z$	4
$L = 2$	$4z^2 + 4z^6$	0	$4 + 4z^4$	0
$L = 3$	$4z^9$	4	$4z^3$	$4z^6$
$L = 4$	$4 + 4z^4 + 4z^8 + 4z^{12}$	0	0	0