Summation of series

Consider the function $\pi \cot \pi z$ on the square C_N with vertices $\pm(N+\frac{1}{2})\pm(N+\frac{1}{2})i$. We have

$$
\begin{aligned}
|\cot \pi z|^2 &= \frac{|\cos \pi z|^2}{|\sin \pi z|^2} \\
&= \frac{\cos^2 \pi x + \sinh^2 \pi y}{\sin^2 \pi x + \sinh^2 \pi y}
\end{aligned}
$$

Along the sides $x = \pm(N + \frac{1}{2})$,

$$
\begin{aligned}
\cos^2 \pi x &= 0 \\
\sin^2 \pi x &= 1 \\
|\cot \pi z|^2 &= \frac{\sinh^2 \pi y}{1 + \sinh^2 \pi y} \\
&= \frac{\sinh^2 \pi y}{\cosh^2 \pi y} \\
|\cot \pi z| &= |\tanh \pi y| < 1 \\
|\pi \cot \pi z| &< \pi < 4
\end{aligned}
$$

Along the sides $y = \pm(N + \frac{1}{2})$,

$$
\begin{aligned}
\cos^2 \pi x + \sinh^2 \pi y &\le 1 + \sinh^2 \pi y = \cosh^2 \pi y \\
\sin^2 \pi x + \sinh^2 \pi y &\ge \sinh^2 \pi y \\
|\cot \pi z|^2 &\le \frac{\cosh^2 \pi y}{\sinh^2 \pi y} \\
|\cot \pi z| &\le |\coth \pi y| \le |\coth \frac{1}{2}\pi| = 1.0903 \\
|\pi \cot \pi z| &< 4
\end{aligned}
$$

Suppose that $f(z)$ is a rational function $O(z^{-2})$ as $z \to \infty$.
For sufficiently large N, all the poles of f lie inside C_N.
Consider

$$\frac{1}{2\pi i} \oint_{C_N} \pi \cot \pi z \, f(z) \, dz$$

On C_N, $|z| > N$, so that $|f| < cN^{-2}$ and

$$\left| \frac{1}{2\pi i} \oint_{C_N} \pi \cot \pi z \, f(z) \, dz \right| \le \frac{1}{2\pi} 4 \frac{c}{N^2} (8N + 4)$$

On the other hand,

$$\frac{1}{2\pi i} \oint_{C_N} \pi \cot \pi z \, f(z) \, dz = \sum Res(\pi \cot \pi z \, f(z))$$

$\pi \cot \pi z$ has simple poles at $z = 0, \pm 1, \ldots, \pm N$, at which its residues are 1.

Assuming there is no overlap with the poles of f, the contribution of these poles to the integral is

$$\sum_{n=-N}^{N} f(n)$$

In addition, we have a contribution S from the poles of f.

Therefore,

$$\sum_{n=-N}^{N} f(n) + S = O(N^{-1})$$

Taking the limit as $N \to \infty$, we obtain

$$\sum_{n=-\infty}^{\infty} f(n) = -S \ .$$

For example, let

$$f(z) = \frac{1}{z^2 + a^2}$$

where a is real.

The poles of f are at $\pm ai$, and the residues of $\pi \cot \pi z \, f(z)$ at those points are;

at $z = ai$;

$$\begin{aligned} \left.\frac{\pi \cot \pi z}{2z}\right|_{z=ia} &= \frac{\pi \cos(i\pi a)}{2ai \sin(i\pi a)} \\ &= \frac{\pi \cosh \pi a}{-2a \sinh \pi a} \\ &= -\frac{\pi}{2a} \coth \pi a \end{aligned}$$

at $z = -ai$

$$\begin{aligned} \left.\frac{\pi \cot \pi z}{2z}\right|_{z=-ia} &= \frac{\pi \cos(-i\pi a)}{-2ai \sin(-i\pi a)} \\ &= \frac{\pi \cosh \pi a}{-2a \sinh \pi a} \\ &= -\frac{\pi}{2a} \coth \pi a \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n=-\infty}^{\infty} \frac{1}{n^2 + a^2} &= \frac{\pi}{a} \coth \pi a \\ \frac{1}{a^2} + 2\sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} &= \frac{\pi}{a} \coth \pi a \\ \sum_{n=1}^{\infty} \frac{1}{n^2 + a^2} &= \frac{1}{2}\left(\frac{\pi}{a} \coth \pi a - \frac{1}{a^2}\right) \end{aligned}$$

When $a = 0$, we have two alternatives.

Firstly, we can consider the third order pole at $z = 0$.

$$\begin{aligned}
\frac{\pi \cot \pi z}{z^2} &= \frac{\pi \cos \pi z}{z^2 \sin \pi z} \\
&= \frac{\pi - \frac{\pi^3 z^2}{2} + \dots}{\pi z^3 - \frac{\pi^3 z^5}{6} + \dots} \\
&= \frac{1}{z^3} \frac{1 - \frac{\pi^2 z^2}{2} + \dots}{1 - \frac{\pi^2 z^2}{6} + \dots} \\
&= \frac{1}{z^3}(1 - \frac{\pi^2 z^2}{2} + \dots)(1 + \frac{\pi^2 z^2}{6} + \dots) \\
&= \frac{1}{z^3}(1 - \frac{\pi^2 z^2}{3} + \dots) \\
&= \frac{1}{z^3} - \frac{\pi^2}{3}\frac{1}{z} + O(z)
\end{aligned}$$

Therefore the residue at $z = 0$ is $-\frac{\pi^2}{3}$, and

$$\begin{aligned}
2\sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{3} \\
\sum_{n=1}^{\infty} \frac{1}{n^2} &= \frac{\pi^2}{6}
\end{aligned}$$

Alternately, we could consider

$$\begin{aligned}
&\lim_{a \to 0} \frac{\pi a \cosh \pi a - \sinh \pi a}{2a^2 \sinh \pi a} \\
&= \lim_{a \to 0} \frac{\pi \cosh \pi a + \pi^2 a \sinh \pi a - \pi \cosh \pi a}{4a \sinh \pi a + 2\pi a^2 \cosh \pi a} \\
&= \lim_{a \to 0} \frac{\pi^3 a \cosh \pi a + \pi^2 \sinh \pi a}{4 \sinh \pi a + 8\pi a \cosh \pi a + 2\pi^3 a^2 \sinh \pi a} \\
&= \lim_{a \to 0} \frac{2\pi^3 \cosh \pi a + O(a^2)}{12\pi \cosh \pi a + O(a^2)} \\
&= \frac{\pi^2}{6}
\end{aligned}$$

Cauchy Expansions

The contours C_N which we use for evaluating sums are a special case of nested contours with the properties:

(i) C_N is in the interior of C_{N+1}

(ii) There is a constant k such that $|z| \geq kN$ on C_N

(iii) There is a constant l such that the length of C_N is at most lN.

Another common example is a set of circles $|z| = aN$.

Suppose that $f(z)$ is a meromorphic funtion with simple poles, regular at $z = 0$, which is bounded on such a set of nested contours; that is $|f(z)| \leq M$ for $z \in C_N$.

Consider

$$\frac{1}{2\pi i} \oint_{C_N} \frac{f(\zeta)}{\zeta(\zeta - z)}\, d\zeta$$

On C_N, $|\zeta| \geq kN$, and if $|z| < \frac{1}{2}kN$,

$$|\zeta - z| \geq |\zeta| - |z| > \frac{1}{2}kN$$

Therefore

$$\left| \frac{1}{2\pi i} \oint_{C_N} \frac{f(\zeta)}{\zeta(\zeta - z)}\, d\zeta \right| < \frac{1}{2\pi} \frac{MlN}{\frac{1}{2}k^2 N^2} = \frac{Ml}{\pi k^2} \frac{1}{N}$$

On the other hand, we can evaluate the integral as the sum of its residues.

There are poles at $\zeta = 0$, $\zeta = z$, and at the poles of f.

At $\zeta = 0$, the residue is

$$-\frac{1}{z} f(0)$$

At $\zeta = z$, the residue is

$$\frac{1}{z} f(z)$$

If $\zeta = \zeta_i$ is a pole of f, then near the pole

$$f(\zeta) = \frac{a_i}{\zeta - \zeta_i} + h_i(\zeta)$$

where h_i is regular in a neighbourhood of ζ_i.

Therefore the integrand is

$$\frac{a_i}{\zeta(\zeta - z)(\zeta - \zeta_i)} + \text{ a regular part}$$

and the residue at ζ_i is

$$\frac{a_i}{\zeta_i(\zeta_i - z)} = -\frac{1}{z} \left(\frac{a_i}{z - \zeta_i} + \frac{a_i}{\zeta_i} \right)$$

Therefore, for $|z| < \frac{1}{2}kN$,

$$\frac{1}{z} \left(f(z) - f(0) - \sum_{i=1}^{m(N)} \left(\frac{a_i}{z - \zeta_i} - \frac{a_i}{\zeta_i} \right) \right) = O(N^{-1})$$

Taking the limit as $N \to \infty$ we obtain the expansion

$$f(z) = f(0) + \sum_{i=1}^{\infty} \left(\frac{a_i}{z - \zeta_i} + \frac{a_i}{\zeta_i} \right)$$

If the poles of f are not simple, it can be shown that if

$$f(\zeta) = \sum_{r=1}^{n} \frac{a_i r}{(\zeta - \zeta_i)^r} + h_i(z) = P_i(\zeta) + h_i(\zeta)$$

then the residue of the integrand at $\zeta = \zeta_i$ is

$$-\frac{1}{z}\left(P(z) - P(0)\right)$$

If f has a pole at the origin, so that

$$f(z) = \sum_{r=-n}^{0} a_r z^r + h(z)$$

we obtain the expansion

$$h(z) = h(0) + \sum_{i=1}^{\infty}(P_i(z) - P_i(0))$$
$$f(z) = \sum_{r=-n}^{0} a_r z^r + \sum_{i=1}^{\infty}(P_i(z) - P_i(0))$$

For example, consider $f(z) = \tan z$, which can be shown to be bounded on the squares C_N with vertices $\pm N\pi \pm N\pi i$.

The function has simple poles at $z = \pm(i + \frac{1}{2})\pi$, with residue

$$\left.\frac{\sin z}{-\sin z}\right|_{z=\zeta_i} = -1$$

Therefore

$$\begin{aligned}\tan z = & \tan 0 + \sum_{i=0}^{\infty}\left(\frac{-1}{z-(i+\frac{1}{2})\pi} - \frac{1}{(i+\frac{1}{2})\pi} + \frac{-1}{z+(i+\frac{1}{2})\pi} + \frac{1}{(i+\frac{1}{2})\pi}\right) \\ = & \sum_{i=0}^{\infty} \frac{2z}{(i+\frac{1}{2})^2\pi^2 - z^2}\end{aligned}$$

Since the convergence is uniform on bounded subsets of $\mathbb{C}$, we can integrate this expansion term by term;

$$\int_0^z \tan t\, dt = \sum_{i=0}^{\infty} \int_0^z \frac{2t}{(i+\frac{1}{2})^2\pi^2 - t^2}\, dt$$
$$-\log(\cos z) = -\sum_{i=0}^{\infty} \log(1 - z^2/(i+\frac{1}{2})^2\pi^2)$$
$$\cos z = \prod_{i=0}^{\infty}\left(1 - \frac{z^2}{(i+\frac{1}{2})^2\pi^2}\right)$$

As a second example, consider $f(z) = \pi \cot \pi z$, which we have shown to be bounded on the squares with vertices $\pm(N+\frac{1}{2}) \pm (N+\frac{1}{2})i$.

This function has a pole at $z = 0$, so that we consider instead

$$h(z) = \pi \cot \pi z - \frac{1}{z}$$

which has poles at $z = \pm 1, \pm 2, \ldots$ with residue 1.

Therefore

$$\begin{aligned} h(z) &= h(0) + \sum_{n=1}^{\infty} \left(\frac{1}{z-n} + \frac{1}{n} + \frac{1}{z+n} - \frac{1}{n} \right) \\ &= \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \\ \pi \cot \pi z &= \frac{1}{z} + \sum_{n=1}^{\infty} \frac{2z}{z^2 - n^2} \end{aligned}$$

Integrating h from 0 to z gives

$$\begin{aligned} \log \left(\frac{\sin \pi z}{\pi z} \right) &= \sum_{n=1}^{\infty} \log \left(1 - \frac{z^2}{n^2} \right) \\ \sin \pi z &= \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2} \right) \end{aligned}$$

The Mittag-Leffler Expansion

Suppose that we are given a sequence $\{\zeta_n\}$ of points in $\mathbb{C}$ such that

$$0 < |\zeta_1| \le |\zeta_2| \le \ldots$$

and the sequence has no finite limit points,

and a corresponding set of polynomials

$$G_n(t) = \sum_{k=1}^{m(n)} a_{nk} t^k$$

then we can construct a meromorphic function having singular points precisely at the points ζ_k and such that the principal parts of the function at its singular points are

$$G_n \left(\frac{1}{z - \zeta_n} \right) .$$

Note that this function is not uniquely defined.

If there are only a finite number of singular points, then we can write down the function

$$f(z) = \sum_{n=1}^{N} G_n \left(\frac{1}{z - \zeta_n} \right)$$

which represents the standard partial fraction expansion of the rational function.

However, in general, if there are a infinite number of singular points the sum

$$\sum_{n=1}^{\infty} G_n \left(\frac{1}{z - \zeta_n} \right)$$

does not converge, and we need to modify the terms appearing in the sum.

To this end we consider a series of positive constants $\{\epsilon_n\}$ whose sum is bounded. For example, we could have $\epsilon_n = cn^{-p}$ for $p > 1$ or $\epsilon_n = cr^n$ for $0 < r < 1$.

Now choose some $R > 0$.
If $|\zeta_n| > R$, then the Taylor series expansion of G_n converges uniformly for $|z| \leq R < |\zeta_n|$.
Therefore, given ϵ_n, there is an integer $N(n)$ such that

$$\left| G_n\left(\frac{1}{z-\zeta_n}\right) - \sum_{k=0}^{N(n)} G_n^{(k)}(0)\frac{z^k}{k!} \right| < \epsilon_n$$

for all $|z| \leq R$.
Denoting the Taylor polynomial by $P_n(z)$, we see that the sum

$$\sum \left(G_n\left(\frac{1}{z-\zeta_n}\right) - P_n(z) \right)$$

is dominated by the convergent sum $\sum \epsilon_n$, and therefore by the Weierstrass test the sum converges absolutely and uniformly for $|z| < R$.
Hence the expansion

$$\sum_{n=1}^{\infty} \left(G_n\left(\frac{1}{z-\zeta_n}\right) - P_n(z) \right)$$

satisfies the stated requirements.

If we want a pole at $z = 0$, then we can add an initial term $G_0(\frac{1}{z})$ at the start of the sum.

The most general form for the expansion is obtained by adding an arbitrary entire function to the expansion.

For example, suppose that we want a function with poles at the points $z = \sqrt{n}, n = 1, 2, \ldots$, with principle parts

$$\frac{1}{z-\sqrt{n}}$$

We have

$$\begin{aligned} \frac{1}{z-\sqrt{n}} &= -\frac{1}{\sqrt{n}}\frac{1}{1-zn^{-1/2}} \\ &= -\frac{1}{\sqrt{n}}\left(1 + \frac{z}{\sqrt{n}} + \frac{z^2}{n} + \ldots\right) \\ \frac{1}{z-\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{z}{n} &= -\frac{1}{n\sqrt{n}}\left(z^2 + \frac{z^3}{\sqrt{n}} + \ldots\right) \\ &= O(n^{-3/2}) \end{aligned}$$

Therefore a suitable function is

$$\sum_{n=1}^{\infty} \left(\frac{1}{z-\sqrt{n}} + \frac{1}{\sqrt{n}} + \frac{z}{n} \right)$$

Weierstrass Infinite Product Expansions

We have seen that if the regular function $f(z)$ has a zero of order n at $z = a$;

$$f(z) = (z-a)^n g(z)\ ;\ g(a) \neq 0$$

then the logarithmic derivative has a simple pole at $z = a$ with residue n;

$$\frac{f'(z)}{f(z)} = \frac{n}{z-a} + \frac{g'(z)}{g(z)}$$

Therefore if $f(z)$ is an entire function with zeros of order n_k at the points $\{\zeta_k\}$, then $f'(z)/f(z)$ is a meromorphic function with poles at $\{\zeta_k\}$ at which the principal parts are

$$G_k = \frac{n_k}{z - \zeta_k}$$

Furthermore, since the zeros of f are isolated, these singular points have no finite accumulation point.

Therefore, using the Mittag-Leffler expansion we can write

$$\frac{f'(z)}{f(z)} = h(z) + \frac{n_0}{z} + \sum_{k=1}^{\infty} \left(\frac{n_k}{z-\zeta_k} - P_k(z) \right)$$

In this expansion, $h(z)$ is an entire function and n_0, which represents the number of zeros of f at $z = 0$, may be zero.

Since this expansion converges uniformly on suitable compact sets, we can integrate it term by term in the form

$$\int_0^z \left(\frac{f'}{f} - \frac{n_0}{t} \right) dt = \int_0^z h(t)\, dt + \sum_{k=1}^{\infty} \int_0^z \left(\frac{n_k}{t-\zeta_k} - P_k(t) \right) dt$$

which gives

$$\log\left(\frac{f(z)}{z^{n_0}} \right) = g(z) + \sum_{k=1}^{\infty} \left(n_k \log\left(1 - \frac{z}{\zeta_k}\right) - p_k(z) \right)$$

where $g = \int h$ and $p_k = \int P_k$.

Taking the exponential we obtain the expansion

$$f(z) = e^{g(z)} z^{n_0} \prod_{k=1}^{\infty} \left(1 - \frac{z}{\zeta_k}\right)^{n_k} e^{-p_k(z)}$$

This expansion is known as a Weierstrass infinite product expansion.

The Gamma Function

The Gamma function has been defined by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1}\, dt$$

for $Re(z) > 0$.

It is extended to the remainder of $\mathbb{C}$ by the functional relationship

$$\Gamma(1+z) = z\Gamma(z) \ .$$

The Gamma function has no zeros, so that the function

$$f(z) = \frac{1}{\Gamma(z)}$$

is entire.

Furtmmermore, from the functional relationship we see that

$$\begin{aligned} f(z) &= zf(z+1) \\ &= z(z+1)f(z+2) \\ &= \ \dots \\ &= z(z+1)(z+2)\dots(z+n)f(z+n+1) \end{aligned}$$

so that f has simple zeros at $\zeta_n = -n$, $n = 0, 1, \dots$.

Therefore the logarithmic derivative has simple poles at these points with residue 1.

Since

$$\sum_{k=1}^\infty \left(\frac{1}{z+k} - \frac{1}{k} \right)$$

converges uniformly on compact sets, we can express

$$\frac{f'}{f} = h(z) + \frac{1}{z} + \sum_{k=1}^\infty \left(\frac{1}{z+k} - \frac{1}{k} \right)$$

and hence

$$f(z) = ze^{g(z)} \prod_{k=1}^\infty \left(1 + \frac{z}{k} \right) e^{-\frac{z}{k}}$$

where the function $g(z)$ needs to be determined.

We have

$$\begin{aligned} \Gamma(z) &= \frac{e^{-g(z)}}{z \prod_{k=1}^\infty \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}} \\ &= \lim_{n\to\infty} F_n(z) \end{aligned}$$

where

$$F_n(z) = \frac{e^{-g(z)}}{z\prod_{k=1}^{n}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}}$$
$$= \frac{n!\exp\left(-g(z)+\sum_{k=1}^{n}\frac{z}{k}\right)}{z(z+1)\dots(z+n)}$$

From the functional relationship we have

$$\begin{aligned}
1 &= \frac{z\Gamma(z)}{\Gamma(z+1)} \\
&= \lim_{n\to\infty}\frac{zF_n(z)}{F_n(z+1)} \\
&= \lim_{n\to\infty}(z+n+1)\exp\left(g(z+1)-g(z)-\sum_{k=1}^{n}\frac{1}{k}\right) \\
&= \lim_{n\to\infty}\left(1+\frac{z+1}{n}\right)\exp\left(g(z+1)-g(z)-\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)\right) \\
&= \exp\left(g(z+1)-g(z)-\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k}-\log n\right)\right) \\
&= \exp(g(z+1)-g(z)-\gamma)
\end{aligned}$$

where γ is Euler's constant; $\gamma \sim 0.577$.

We also have

$$\begin{aligned}
1 &= \Gamma(1) \\
&= \lim_{n\to\infty}F_n(1) \\
&= \lim_{n\to\infty}\frac{\exp\left(-g(1)+\sum_{k=1}^{n}\frac{1}{k}\right)}{n+1} \\
&= \lim_{n\to\infty}\frac{\exp\left(-g(1)+\sum_{k=1}^{n}\frac{1}{k}-\log n\right)}{1+\frac{1}{n}} \\
&= \exp(-g(1)+\gamma)
\end{aligned}$$

The function $g(z) = \gamma z$ obviously satisfies these two conditions, and it can be shown that this is the correct choice.

Therefore

$$\Gamma(z) = \frac{e^{-\gamma z}}{z\prod_{k=1}^{\infty}\left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}}$$

Since

$$\Gamma(-z) = \frac{e^{\gamma z}}{(-z)\prod_{k=1}^{\infty}\left(1-\frac{z}{k}\right)e^{\frac{z}{k}}}$$

we have

$$\begin{aligned}
\Gamma(z)\Gamma(-z) &= \frac{1}{(-z)z\prod_{k=1}^{\infty}\left(1-\frac{z^2}{k^2}\right)} \\
&= \frac{\pi}{(-z)\sin\pi z} \\
\Gamma(z)(-z)\Gamma(-z) &= \Gamma(z)\Gamma(1-z) \\
&= \frac{\pi}{\sin\pi z}
\end{aligned}$$

In particular,

$$\begin{aligned}
\Gamma\left(\frac{1}{2}\right)^2 &= \frac{\pi}{\sin\frac{1}{2}\pi} = \pi \\
\Gamma\left(\frac{1}{2}\right) &= \sqrt{\pi}
\end{aligned}$$