Power series expansions

Many complex valued functions are represented by power series expansions.

We begin this section by recalling some of the results from earlier years about infinite series.

Convergence tests

The comparison test.

If $|a_n| \leq b_n$ for all $n > N$ for some N, and if $\sum b_n$ converges, then $\sum a_n$ converges absolutely.

The ratio test.

(a) If $\lim_{n\to\infty} |a_{n+1}|/|a_n| < 1$. then $\sum a_n$ converges absolutely.

If the limit is > 1, then the series diverges. If the limit $= 1$, or if the limit does not exist, the test is inconclusive.

(b) $\sum a_n(z - z_0)^n$ converges absolutely if

$$\lim_{n\to\infty} \frac{|a_{n+1}||z - z_0|^{n+1}}{|a_n||z - z_0|^n} < 1$$
$$|z - z_0| < \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

providing the limit exists.

When it does,

$$R = \lim_{n\to\infty} \frac{|a_n|}{|a_{n+1}|}$$

is called the **radius of convergence** of the power series.

Cauchy's root test.

(a) if $\limsup |a_n|^{1/n} = q < 1$, then $\sum a_n$ converges absolutely.

If $q > 1$, the series diverges. If $q = 1$, the test is inconclusive.

(b) $\sum a_n(z - z_0)^n$ converges absolutely if

$$\limsup |a_n|^{1/n}|z - z_0| < 1$$
$$|z - z_0| < \frac{1}{\limsup |a_n|^{1/n}}$$

In this case, $\limsup |a_n|^{1/n}$ always exists, and the radius of convergence is

$$R = \frac{1}{\limsup |a_n|^{1/n}}$$

However, Cauchy's test is usually more dofficult to apply than the ratio test.

The Geometric Progression.

Consider

$$a + ar + ar^2 + \cdots + ar^{n-1}$$
$$= \sum_{k=0}^{n-1} ar^k$$

$$\begin{aligned}
\text{If} \quad S &= a + ar + ar^2 + \cdots + ar^{n-1} \\
\text{then} \quad rS &= \quad ar + ar^2 + \cdots + ar^{n-1} + ar^n \\
S - rS = (1-r)S &= a - ar^n = a(1-r^n) \\
S &= a\frac{1-r^n}{1-r}
\end{aligned}$$

If $|r| < 1$, then $r^n \to 0$ as $n \to \infty$.

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

If $|\zeta - z_0| < |z - z_0|$,

$$\begin{aligned}
\frac{1}{z-\zeta} &= \frac{1}{(z-z_0) - (\zeta - z_0)} \\
&= \frac{1}{z-z_0}\frac{1}{1 - \frac{\zeta - z_0}{z-z_0}} \\
&= \sum_{k=0}^{\infty} \frac{1}{z-z_0}\left(\frac{\zeta - z_0}{z - z_0}\right)^k \\
&= \sum_{k=0}^{\infty} \frac{(\zeta - z_0)^k}{(z-z_0)^{k+1}}
\end{aligned}$$

If $|\zeta - z_0| > |z - z_0|$,

$$\begin{aligned}
\frac{1}{z-\zeta} &= -\frac{1}{\zeta - z} \\
&= -\sum_{k=0}^{\infty} \frac{(z-z_0)^k}{(\zeta - z_0)^{k+1}} \\
&= -\sum_{l=-1}^{-\infty} \frac{(\zeta - z_0)^l}{(z-z_0)^{l+1}}
\end{aligned}$$

Taylor's Theorem

Suppose that $f(z)$ is regular inside and on the circle $C \; : \; |z - z_0| = R$.
For any ζ inside the circle,

$$|\zeta - z_0| < |z - z_0|$$

when z is on C, and Cauchy's Integral Formula gives

$$\begin{aligned}
f(\zeta) &= \frac{1}{2\pi i}\oint_C \frac{f(z)}{z-\zeta}\,dz \\
&= \frac{1}{2\pi i}\oint_C f(z)\sum_{k=0}^{\infty}\frac{(\zeta - z_0)^k}{(z-z_0)^{k+1}}\,dz \\
&= \sum_{k=0}^{\infty}(\zeta - z_0)^k\left(\frac{1}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{k+1}}\,dz\right) \\
&= \sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(\zeta - z_0)^k
\end{aligned}$$

Let $M = \max|f(z)|$ on C.
Using Cauchy's inequalities,

$$\left|f^{(k)}(z_0)\right| \le \frac{k!M}{R^k}$$

so that

$$\begin{aligned}\left|\sum_{k=0}^{\infty}\frac{f^{(k)}(z_0)}{k!}(\zeta - z_0)^k\right| &\le \sum_{k=0}^{\infty}\left|\frac{f^{(k)}(z_0)}{k!}\right||\zeta - z_0|^k \\ &\le \sum_{k=0}^{\infty} M\left(\frac{|\zeta - z_0|}{R}\right)^k\end{aligned}$$

which converges absolutely for $|\zeta - z_0| < R$ (at least).

For given z_0, the radius of convergence of the Taylor series expansion is determined by the size of the largest circle within which $f(z)$ remains regular.
That is, the radius of convergence is the distance from z_0 to the nearest singular point of the function.

A function possessing convergent Taylor series expansions about every point in a domain is said to be **analytic**.
This result shows that a regular function is analytic; i.e. in the complex plane, the existence of the derivative at every point of a domain is sufficient to guarantee the existence of derivatives of all orders, and the convergence of the Taylos series to the function in the domain.

e.g.

$$\begin{aligned} f(z) &= \log z \\ f'(z) = \frac{1}{z}\ ;\ f''(z) &= -\frac{1}{z^2}\ ;\ f'''(z) = \frac{2}{z^3} \\ f^{(k)}(z) &= (-1)^{k-1}\frac{(k-1)!}{z^k} \\ \log z = \log z_0 &+ \sum_{k=1}^{\infty}(-1)^{k-1}\frac{(k-1)!}{k!z_0^k}(z - z_0)^k \\ &= \log z_0 - \sum_{k=1}^{\infty}\frac{1}{k}\left(1 - \frac{z}{z_0}\right)^k \end{aligned}$$

This series converges provided

$$\left|1 - \frac{z}{z_0}\right| < 1$$

that is

$$|z - z_0| < |z_0|$$

e.g.

$$f(z) = (1+z)^p \quad f(0) = 1$$
$$f'(z) = p(1+z)^{p-1} \quad f'(0) = p$$
$$f"(z) = p(p-1)(1+z)^{p-2} \quad f"(0) = p(p-1)$$
$$f^{(n)}(0) = p(p-1)\dots(p-(n-1))$$
$$f(z) = \sum_{n=0}^{\infty} \frac{p(p-1)\dots(p-n+1)}{n!} z^n$$

This is the Generalised Binomial Expansion.

If p is a positive integer, the series terminates, and we have the well-known polynomial expansion.

Otherwise,

$$\frac{a_n}{a_{n+1}} = \frac{p(p-1)\dots(p-n+1)}{n!} \frac{(n+1)!}{p(p-1)\dots(p-n)} = \frac{n+1}{p-n}$$
$$\left|\frac{a_n}{a_{n+1}}\right| \to 1 \text{ as } n \to \infty$$

Therefore the series converged for $|z| < 1$.

Analytic Functions

If

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

converges for $|z - z_0| < R$, then the series can be differentiated term by term within the circle of convergence, and

$$f'(z) = \sum_{k=1}^{\infty} k a_k (z - z_0)^{k-1}$$

and the derived series has the same radius of convergence as the original series.

This result shows that an analytic function is regular.

We have now shown that in the complex plane, regular, holomorphic and analytic functions are the same. For this reason the labels are used interchangeably.

When

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_0)}{k!} (z - z_0)^k$$

for $|z - z_0| < R_0$, then

$$f'(z) = \sum_{k=1}^{\infty} \frac{f^{(k)}(z_0)}{(k-1)!} (z - z_0)^{k-1}$$

and

$$f^{(n)}(z) = \sum_{k=n}^{\infty} \frac{f^{(k)}(z_0)}{(k-n)!} (z - z_0)^{k-n}$$

also for $|z - z_0| < R_0$.

If

$$
\begin{aligned}
f(z) &= \sum_{k=0}^{\infty} a_k (z-z_0)^k \\
&\text{then} \\
f'(z) &= \sum_{k=1}^{\infty} k a_k (z-z_0)^{k-1} \\
f''(z) &= \sum_{k=2}^{\infty} k(k-1)(z-z_0)^{k-2} \\
f^{(n)}(z) &= \sum_{k=n}^{\infty} k(k-1)\ldots(k-(n-1))(z-z_0)^{k-n} \\
f(z_0) &= a_0 \quad a_0 = f(z_0) \\
f'(z_0) &= 1a_1 \quad a_1 = \frac{f'(z_0)}{1!} \\
f''(z_0) &= 2.1a_2 \quad a_2 = \frac{f''(z_0)}{2!} \\
f^{(n)}(z_0) &= n.(n-1)\ldots 1a_n \quad a_n = \frac{f^{(n)}(z_0)}{n!}
\end{aligned}
$$

This shows that *ANY* convergent power series is the Taylor series.

Therefore we can use a variety of methods to obtain Taylor series expansions.

e.g. Starting with the expansion for e^z, we have

$$
e^z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k
$$

$$
\begin{aligned}
e^{-z^2} &= \sum_{k=0}^{\infty} \frac{1}{k!}(-z^2)^k \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} z^{2k}
\end{aligned}
$$

$$
\begin{aligned}
\operatorname{erf}(z) &= \frac{2}{\sqrt{\pi}} \int_0^z e^{-s^2}\, ds \\
&= \frac{2}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{z^{2k+1}}{2k+1}
\end{aligned}
$$

Similarly, starting from the Binomial expansion, we get

$$(1+z)^p = 1 + pz + \frac{p(p-1)}{2!}z^2 + \dots$$

$$\begin{aligned}
(1-z^2)^{-1/2} &= 1 + \left(-\frac{1}{2}\right)(-z^2) \\
&\quad + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-z^2\right)^2 + \dots \\
&\quad + \frac{1}{k!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\dots\left(-\frac{2p-1}{2}\right)\left(-z^2\right)^k + \dots \\
&= 1 + \frac{1}{2}z^2 + \frac{1}{2!}\frac{1}{2}\frac{3}{2}z^4 + \dots \\
&\quad + \frac{1}{k!}\frac{1}{2}\frac{3}{2}\dots\frac{2k-1}{2}z^{2k} + \dots \\
&= 1 + \frac{1}{2}z^2 + \frac{1.3}{2^2 2!}z^4 + \dots \\
&\quad + \frac{1.3\dots(2k-1)}{2^k k!}z^{2k} + \dots \\
&= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2} z^{2k}
\end{aligned}$$

$$\begin{aligned}
\arcsin z &= \int_0^z \frac{ds}{\sqrt{1-s^2}} \\
&= \sum_{k=0}^{\infty} \frac{(2k)!}{2^{2k}(k!)^2}\frac{z^{2k+1}}{2k+1}
\end{aligned}$$

Laurent's Theorem

Suppose that $f(z)$ is regular in the annular region bounded by $C_1 \ : \ |z-z_0| = R_1$ and $C_2 \ : \ |z-z_0| = R_2$, where $R_1 > R_2$.

If ζ is in the annulus, then $|\zeta - z_0| < |z - z_0|$ when z is on C_1, and $|\zeta - z_0| > |z - z_0|$ when z is on C_2.

Applying Cauchy's Integral Formula to the annular region we have

$$\begin{aligned}
f(\zeta) &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{z-\zeta} - \frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{z-\zeta} \\
&= \sum_{k=0}^{\infty} (z-z_0)^k \left(\frac{1}{2\pi i}\oint_{C_1} \frac{f(z)}{(z-z_0)^{k+1}}\,dz\right) \\
&\quad + \sum_{l=-1}^{-\infty} (z-z_0)^l \left(\frac{1}{2\pi i}\oint_{C_2} \frac{f(z)}{(z-z_0)^{l+1}}\,dz\right) \\
&= \sum_{-\infty}^{\infty} a_n (z-z_0)^n
\end{aligned}$$

This expansion is known as a *Laurent expansion*.

For given z_0, the expansion is unique for the particular annular region under consideration, but the same function may have different expansions in different annuli about z_0.

For example, consider the function

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1} \; .$$

For $|z| < 1$, we have the Taylor series expansions

$$\begin{aligned} f(z) &= \frac{1}{1-z} - \frac{1}{2}\frac{1}{1-z/2} \\ &= \sum_{n=0}^{\infty} z^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n \\ &= \sum_{n=0}^{\infty}\left(1 - 2^{-(n+1)}\right) z^n \end{aligned}$$

For $1 < |z| < 2$, we can retain the previous expansion for $1/(z-2)$, but we need to replace the expansion of $1/(z-1)$ with one which is valid for $|z| > 1$.

$$\begin{aligned} f(z) &= -\frac{1}{z}\frac{1}{1-1/z} - \frac{1}{2}\frac{1}{1-z/2} \\ &= -\sum_{n=0}^{\infty} z^{-(n+1)} - \sum_{n=0}^{\infty} 2^{-(n+1)} z^n \end{aligned}$$

Finally, for $|z| > 2$, we have

$$\begin{aligned} f(z) &= -\frac{1}{z}\frac{1}{1-1/z} + \frac{1}{z}\frac{1}{1-2/z} \\ &= -\sum_{n=0}^{\infty} z^{-(n+1)} + \sum_{n=0}^{\infty} 2^n z^{-(n+1)} \\ &= \sum_{n=0}^{\infty} \left(2^n - 1\right) z^{-(n+1)} \end{aligned}$$

The Laurent expansion represents the function f as the sum of two functions; $f = g + h$; where

$$g(z) = \sum_{k=0}^{\infty}\left(\frac{1}{2\pi i}\oint_{C_1}\frac{f(z)}{(z-z_0)^{k+1}}\,dz\right)(z-z_0)^k$$

is regular for $|z - z_0| < R_1$,

and

$$h(z) = \sum_{k=1}^{\infty}\left(\frac{1}{2\pi i}\oint_{C_2} f(z)(z-z_0)^{k-1}\,dz\right)\frac{1}{(z-z_0)^k}$$

is regular for $|z - z_0| > R_2$.

Singular Points

Suppose that the function f has an isolated singular point (where it is not regular) at z_0, and consider the local Laurent expansion

$$f(z) = \sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k} + \sum_{k=0}^{\infty} a_k (z-z_0)^k$$

for $0 < |z-z_0| < R$.

We classify the singular point at z_0 according to the form of the principal part

$$\sum_{k=1}^{\infty} a_{-k}(z-z_0)^{-k}$$

of this expansion.

If this part of the expansion vanishes identically, we say that the singularity at z_0 is removable.

For example

$$\frac{\sin z}{z} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n+1)!}$$

has a removable singularity at $z = 0$.

If $f(z)$ has a removable singularity, then $f(z)$ is bounded in a neighbourhood of z_0.

Conversely, If $f(z)$ is bounded by M in a neighbourhood of z_0, by choosing a contour $|z-z_0| = \epsilon$ lying within this neighbourhood, we have

$$\begin{aligned} |a_{-k}| &= \frac{1}{2\pi} \left| \oint f(z)(z-z_0)^{k-1}\, dz \right| \\ &\leq \frac{1}{2\pi} M\epsilon^{k-1} 2\pi\epsilon = M\epsilon^k \end{aligned}$$

so that the singularity is removable.

It is usual in this cases to remove the singularity at z_0 by defining

$$f(z_0) = a_0 \ .$$

For example,

$$f(z) = \begin{cases} \frac{e^z-1}{z} & z \neq 0 \\ 1 & z = 0 \end{cases}$$

is an entire function whose Taylor series expansion about $z = 0$ is

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{(n+1)!}$$

If $a_{-m} \neq 0$, but $a_{-k} = 0$ for all $k > m$, we say that f has a pole of order m at z_0.

If $f(z)$ has a pole of order m at z_0, then $f(z) = g(z)/(z-z_0)^m$, where $g(z)$ is regular in a neighbourhood of z_0, and $g(z_0) = a_{-m} \neq 0$.

Therefore there is a neighbourhood of z_0 in which $|g(z)| > \frac{1}{2}|a_{-m}|$, and within this neighbouthood

$$|f(z)| > \frac{1}{2} \frac{|a_{-m}|}{|z-z_0|^m}$$

which diverges to infinity as $z \to z_0$.

Finally, if the principal part has an infinite number of non-zero terms in the expansion, $f(z)$ has an essential singularity at z_0.

Consider any complex number A and any real $\epsilon > 0$.

Suppose that $|f(z) - A| > \epsilon$ in any neighbourhood of z_0.

Then

$$\frac{1}{|f(z) - A|} < \frac{1}{\epsilon}$$

in this neighbourhood, so that $g(z) = 1/(f(z) - A)$ has a removable singularity at z_0.

Therefore

$$f(z) = A + \frac{1}{g(z)}$$

which has at worst a pole at z_0 if $g(z_0) = 0$.

Therefore our supposition is not valid.

If $f(z)$ has an essential singularity at z_0, then $f(z)$ comes arbitrarily close to every complex number in every neighbourhood of z_0.

In fact it can be shown that, with possibly two exceptions, $f(z)$ takes every value in $\mathbb{C}$ in every neighbourhood of z_0

For example, the function $e^{1/z}$ has an essential singularity at $z = 0$.

For any $A \neq 0$, we can solve

$$\begin{aligned} e^{1/z} &= A \\ \frac{1}{z} &= \log(A) = \log|A| + i\arg(A) \\ &= \log|A| + i(\theta + 2n\pi) \quad \text{for some } \theta \\ z &= \frac{\log|A| - i(\theta + 2n\pi)}{(\log|A|)^2 + (\theta + 2n\pi)^2} \end{aligned}$$

and for any $\epsilon > 0$ we can find a solution (in fact infinitely many of them) with $|z| < \epsilon$ by taking n sufficiently large.

Analytic continuation

Suppose that $f(z)$ is an analytic function, such that

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \text{for} \quad |z - z_0| < R_0 \ .$$

For any z_1 such that $|z_1 - z_0| < R_0$, we can use the power series and its derivatives to determine $f(z_1)$, $f'(z_1)$, etc., and hence we can (in theory at least) determine the coefficients in the expansion

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_1)}{k!} (z - z_1)^k$$

which converges for $|z - z_1| < R_1$.

If $R_1 = R_0 - |z_1 - z_0|$, this new series gives no further information about the function, but if $R_1 > R_0 - |z_1 - z_0|$, then this expansion extends the region in which we can evaluate f.

This extension is called an *analytic continuation* of f.

Zeros of analytic functions

If $a_0 = a_1 = \cdots = a_{n-1} = 0$, but $a_n \neq 0$, then

$$f(z) = \sum_{k=0}^{\infty} a_k (z - z_0)^k$$

has a *zero of order n* at z_0.

$f(z) = (z - z_0)^n g(z)$, where

$$g(z) = \sum_{k=0}^{\infty} a_{k+n} (z - z_0)^k \ .$$

$g(z_0) = a_n \neq 0$. The function g is continuous.
Therefore given $\epsilon = \frac{1}{2}|a_n|$, there is $\delta > 0$ such that

$$|g(z) - a_n| < \frac{1}{2}|a_n| \quad \text{for} \quad |z - z_0| < \delta$$
$$|g(z)| > \frac{1}{2}|a_n|$$

Therefore there is a neighbourhood $|z - z_0| \leq r$ in which $g(z)$ is not zero. This shows that the zeros of f are isolated.
In this neighbourhood, $g'(z)/g(z)$ is regular.
Suppose that $g(z) \neq 0$ for $|z - z_0| \leq r$ at least.
Consider

$$\begin{aligned} &\frac{1}{2\pi i} \oint_{|z-z_0|=r} \frac{f'(z)}{f(z)}\, dz \\ = &\frac{1}{2\pi i} \oint \frac{n(z-z_0)^{n-1} g(z) + (z-z_0)^n g'(z)}{(z-z_0)^n g(z)}\, dz \\ &= \frac{1}{2\pi i} \oint \left(\frac{n}{z - z_0} + \frac{g'(z)}{g(z)} \right) dz \\ &= n + 0 \end{aligned}$$

which is the number of zeros of $f(z)$ inside this contour.

Now consider
$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz$$
for an arbitrary scroc C, where f is regular inside and on C, and such that no zero of f lies on C.

As a consequence of Cauchy's theorem, we can replace C by a series of circles around the *isolated* zeros of f inside C.

Hence
$$\frac{1}{2\pi i} \oint_C \frac{f'(z)}{f(z)}\, dz = \text{ the number of zeros of } f \text{ inside } C$$

On the other hand, if we choose an arbitrary starting point a on C,

$$\begin{aligned}\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz &= \frac{1}{2\pi i}\,\log(f(z))\big|_a^a \\ &= \frac{1}{2\pi i}(\log|f(a)| + i\arg(f(a)) \\ &\quad - \log|f(a)| - i\arg(f(a)))\end{aligned}$$

While $\log|f(a)|$ takes the same value at the start and the finish of the contour, $\arg(f(a))$ increases by a factor 2π each time the curve $f(z)$ circles the origin in a counter-clockwise direction as z moves along C.

Hence

$$\frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)}\,dz = \frac{1}{2\pi}\ Var\ \ \arg f(z)|_C$$

which is the number of times $f(z)$ circles the origin as z goes around C.

This means that to determine the number of zeros of an analytic function $f(z)$ inside a closed curve $z = C(t)$; $a \le t \le b$, we plot the graph of $f(C(t))$ for $a \le t \le b$, and count how many times it circles the origin.

Rouché's Theorem

Suppose that $f(z)$ and $g(z)$ are regular inside and on the scroc C.

Then, if $|g(z)| < |f(z)|$ on C, $f(z)$ and $f(z) + g(z)$ have the same number of zeros inside C.

Proof. The number of zeros of $f + g$ inside C is equal to

$$\frac{1}{2\pi} Var\ \arg(f+g)\Big|_C \quad .$$

However

$$\begin{aligned} f + g &= f\left(1 + \frac{g}{f}\right) \\ \arg(f+g) &= \arg(f) + \arg\left(1 + \frac{g}{f}\right) \\ Var\arg(f+g)|_C &= Var\arg(f)|_C + Var\arg\left(1+\frac{g}{f}\right)\Big|_C \end{aligned}$$

and on C, $|g/f| < 1$, so that $1 + g/f$ remains inside the unit circle centered at 1 as z traverses C.

Therefore,

$$Var\arg\left(1 + \frac{g}{f}\right)\Big|_C = 0$$

and

$$Var\arg(f+g)|_C = Var\arg(f)|_C$$

as required.

e.g. Consider

$$z^7 - 5z^3 + 12$$

On $C : |z| = 1$, choose $f(z) = 12$ and $g(z) = z^7 - 5z^3$.

$$\left|z^7 - 5z^3\right| \le |z|^7 + 5|z|^3 = 6 < 12$$

on the unit circle, so that Rouché's Theorem holds, and $z^7 - 5z^3 + 12$ has as many zeros inside the unit circle as 12, namely none.

On $C : |z| = 2$, choose $f(z) = z^7$, and $g(z) = 12 - 5z^3$.

On C, $|z^7| = 2^7 = 128$, and

$$|12 - 5z^3| \le 12 + 5|z|^3 = 12 + 40 = 52 < 128$$

so that $z^7 - 5z^3 + 12$ has all 7 of its zeros inside $|z| = 2$.

Stability

The stability of linear dynamical systems depends on having all the roots of a certain polynomial either

(a) In the left half complex plane for continuous systems

(b) Inside the unit circle for discrete systems.

For the second case, we can use the preceding theory as is.

For the first case, we need a minor modification.

Since we are dealing with a polynomial, there are only a finite number of zeros, and therefore for sufficiently large R, all the roots lie inside the circle $|z| = R$.

All the roots of the polynomial will lie in the left half plane if they lie inside the semicircle bounded by the imaginary axis and the circular arc $z = Re^{i\theta}$, $\frac{\pi}{2} \le \theta \le \frac{3\pi}{2}$.

Given the polynomial

$$f(z) = z^n + a_{n-1}z^{n-1} + \dots a_o$$

we can write this as

$$f(z) = z^n \left(1 + a_{n-1}z^{-1} + \dots a_0 z^{-n}\right)$$

and for $|z|$ suffuciently large

$$f(z) = z^n(1 + \delta(z)) \text{ where } |\delta| << 1$$

Therefore, on the semicircular arc,

$$\begin{aligned} f(z) &\sim z^n = R^n e^{in\theta} \\ \arg(f(z) &\sim n\theta \; ; \; \frac{\pi}{2} \le \theta \le \frac{3\pi}{2} \\ Var(\arg(f(z)) &\sim n\pi \end{aligned}$$

where the approximations approach equality as $R \to \infty$.

If all the zeros of f lie in the left half plane, then the total variation in $\arg(f(z)$ around the contour is $2n\pi$.

Therefore, in this case the variation in $\arg(f(z)$ along the imaginary axis is also $n\pi$.

If, in addition, the coefficients a_i of the polynomial are real, so that $f(x)$ is a real polynomial, then by symmetry the variation of the argument along the positive imaginary axis will be half the variation along the whole imaginary axis.

Therefore, for such a "real" polynomial, all the roots will have negative real part provided the variation of the argument of $f(iy)$ as y varies from 0 to ∞ is $\frac{1}{2}n\pi$.

This variation can be determined by plotting the graph of

$$(Re(f(iy)), Im(f(iy))) \ ; \ y > 0 \ .$$

For example, if

$$f(z) = z^3 + 6z^2 + 11z + 6$$
$$f(iy) = -iy^3 - 6y^2 + i11y + 6$$

and all the roots of this polynomial will lie in the left half plane provided the graph

$$\left(6 - 6y^2, 11y - y^3\right) \ ; \ y > 0$$

goes three-quarters of the way around the origin.

When $y = 0$, the graph is at $(6, 0)$.

When $y = 1$, the graph is at $(0, 10)$.

When $y = \sqrt{11}$, the graph is at $(-60, 0)$.

Furthermore, these are the only places where the graph crosses the axes.

As $y \to \infty$, the graph goes to ∞ in the third quadrant.

Therefore the variation in the argument, which must be an integral multiple of $\frac{1}{2}\pi$, is $\frac{3}{2}\pi$.

This shows that all the roots of this polynomial lie in the left half plane.

(They are in fact -1, -2 and -3.)

This result is known in the Engineering literature as *Nyquist's criterion*.

Lagrange's Expansion

Lemma. *If the analytic function $w(t)$ has a simple zero at $t = \alpha$ inside the scroc C and no other zeros inside C, and if $F(z)$ is analytic inside and on C, then*

$$\frac{1}{2\pi i} \oint_C F(t) \frac{w'(t)}{w(t)} \, dt \ = \ F(\alpha) \ .$$

Suppose that z is defined implicitly as a function of ζ by the equation

$$z = a + \zeta\phi(z) \ ,$$

where $\phi(z)$ is an analytic function of z inside and on a circle C: $|z - a| = \rho$, and $\phi(a) \neq 0$.

If $M = \max|\phi|$ on C, then by Rouché's Theorem the equation has a unique solution inside C for $|\zeta| < \rho/M$.

The function $w(t) = t - a - \zeta\phi(t)$ has a simple zero at $t = z(\zeta)$. Therefore, for any function F, analytic inside and on C, and for $|\zeta| < \rho/M$,

$$\begin{aligned}
F(z) &= \frac{1}{2\pi i}\oint_C F(t)\frac{1-\zeta\phi'(t)}{t-a-\zeta\phi(t)}\,dt \\
&= \frac{1}{2\pi i}\oint_C \frac{F(t)}{t-a}(1-\zeta\phi'(t))\frac{1}{1-\zeta\phi(t)/(t-a)}\,dt \\
&= \frac{1}{2\pi i}\oint_C \frac{F(t)}{t-a}(1-\zeta\phi'(t))\left(\sum_{n=0}^{\infty}\zeta^n\left(\frac{\phi(t)}{t-a}\right)^n\right)\,dt \\
&\text{(since } |\zeta\phi/(t-a)| < (\rho/M)M/\rho = 1 \text{ on } C) \\
&= \frac{1}{2\pi i}\oint_C \frac{F(t)}{t-a}\,dt \\
&\quad + \sum_{n=1}^{\infty}\zeta^n\frac{1}{2\pi i}\oint_C F(t)\left(\frac{\phi^n}{(t-a)^{n+1}} - \frac{\phi^{n-1}\phi'}{(t-a)^n}\right)\,dt \\
&= F(a) - \sum_{n=1}^{\infty}\zeta^n\frac{1}{2\pi i}\oint_C \frac{F(t)}{n}\left(\frac{d}{dt}\left(\frac{\phi^n}{(t-a)^n}\right)\right)\,dt \\
&= F(a) + \sum_{n=1}^{\infty}\frac{\zeta^n}{n!}\frac{(n-1)!}{2\pi i}\oint_C \frac{F'(t)\phi^n(t)}{(t-a)^n}\,dt \\
&\quad \text{(integrating by parts)} \\
&= F(a) + \sum_{n=1}^{\infty}\frac{\zeta^n}{n!}\left[\frac{d^{n-1}}{dt^{n-1}}\left(F'(t)\phi^n(t)\right)\right]_{t=a} \\
&\quad \text{Cauchy's Integral Theorem}
\end{aligned}$$

This expansion is known as Lagrange's expansion.

In particular, if $F(z) = z$,

$$z = a + \sum_{n=1}^{\infty}\frac{\zeta^n}{n!}D^{n-1}\phi^n(a)$$

We can use this expansion to determine inverse functions.

Suppose $w = f(z)$, and $f'(z_0) \neq 0$. Then locally we have

$$\begin{aligned}
w - w_0 &= f(z) - f(z_0) \\
&= \frac{f(z)-f(z_0)}{z-z_0}(z-z_0) \\
&= \psi(z)(z-z_0) \quad \text{where} \quad \psi(z_0) = f'(z_0) \neq 0 \\
z - z_0 &= (w-w_0)\phi(z) \quad \text{where} \quad \phi(z) = 1/\psi(z) \\
&= \sum_{n=1}^{\infty}\frac{(w-w_0)^n}{n!}\,D^{n-1}\phi^n(t)\big|_{t=z_0}
\end{aligned}$$

For example,

$$
\begin{aligned}
\text{if} \qquad w &= z - \frac{1}{z} \; ; \; z_0 = 1 \\
&= \frac{z^2 - 1}{z} \\
&= \frac{z+1}{z}(z-1) \\
z - 1 &= w\left(\frac{z}{z+1}\right) \\
&= \sum_{n=1}^{\infty} \frac{w^n}{n!} \, D^{n-1}\left(\frac{t^n}{(t+1)^n}\right)\Bigg|_{t=1} \\
z &= 1 + \frac{1}{2}w + \frac{1}{8}w^2 + \dots
\end{aligned}
$$