

INTEGRATION IN THE COMPLEX PLANE
CONTOUR INTEGRATION

Consider the expression

$$\int_{z_0}^{z_1} f(z) dz .$$

What meaning can we attach to it in the complex plane?

When we evaluate a real integral

$$\int_{x_0}^{x_1} f(x) dx$$

there is a unique path from x_0 to x_1 .

However, in the complex plane there are infinitely many ways of going from z_0 to z_1 . Therefore, in defining the integral we have to specify the path - 'contour' - along which the integral is to be evaluated.

We will take as our contours of integration piecewise smooth paths in \mathbb{C} ;

$$z(t) = x(t) + iy(t) ; \alpha \leq t \leq \beta ; z(\alpha) = z_0 , z(\beta) = z_1$$

where the functions $x(t)$ and $y(t)$ are piecewise differentiable functions of t . For such a path the integral

$$\int_{z_0}^{z_1} f(z) dz$$

is now defined in terms of real integrals by

$$\begin{aligned} \int_{z_0}^{z_1} f(z) dz &= \\ \int_{\alpha}^{\beta} (u(x(t), y(t)) + iv(x(t), y(t))) (x'(t) + iy'(t)) dt \\ &= \int_{\alpha}^{\beta} (ux' - vy') dt + i \int_{\alpha}^{\beta} (uy' + vx') dt . \end{aligned}$$

One particular class of contour which appears regularly consists of curves which are closed and which do not intersect themselves. For such a curve we choose $z_0 = z_1$ arbitrarily on the curve. The positive orientation of the curve is when it is traversed in the anti-clockwise direction.

Such a contour is referred to as a *simple, closed, rectifiable, oriented curve*, or **scroc** for short.

Since the terminal points of the path of integration are indeterminate, we denote the integral over a scroc by

$$\oint_C f(z) dz .$$

If C is a scroc, then $-C$ represents the same curve traversed in the clockwise direction, and

$$\oint_{-C} f(z) dz = - \oint_C f(z) dz .$$

e.g.

$$\int_0^{1+i} \bar{z} dz$$

(i) The contour is the straight line joining 0 and $1 + i$.

$$\begin{aligned} z &= (1 + i)t ; 0 \leq t \leq 1 \\ dz &= (1 + i)dt \\ \bar{z} &= x - iy = t - it \\ \int_0^{1+i} \bar{z} dz &= \int_0^1 t(1 - i)(1 + i) dt \\ &= \int_0^1 2t dt = t^2 \Big|_0^1 = 1 \end{aligned}$$

(ii) The contour consists of the straight line segments

$$\begin{aligned} y &= 0 ; 0 \leq x \leq 1 \\ x &= 1 ; 0 \leq y \leq 1 \end{aligned}$$

On the first part of the contour,

$$z = x ; dz = dx ; \bar{z} = x$$

and the contribution to the integral is

$$\int_0^1 x dx = \frac{1}{2}x^2 \Big|_0^1 = \frac{1}{2}$$

On the second part of the contour

$$z = 1 + iy ; dz = idy ; \bar{z} = 1 - iy$$

and the contribution to the integral is

$$\int_0^1 (1 - iy) idy = iy + \frac{1}{2}y^2 \Big|_0^1 = i + \frac{1}{2}$$

Combining these two results we obtain

$$\int_0^{1+i} \bar{z} dz = 1 + i$$

(iii) The contour is the parabola $y = x^2$.

$$\begin{aligned}
z &= t + it^2 ; 0 \leq t \leq 1 \\
dz &= (1 + 2it) dt \\
\bar{z} &= x - iy = t - it^2 \\
\int_0^{1+i} \bar{z} dz &= \int_0^1 (t - it^2)(1 + 2it) dt \\
&= \int_0^1 (t + it^2 + 2t^3) dt \\
&= \left. \frac{1}{2}t^2 + \frac{i}{3}t^3 + \frac{1}{2}t^4 \right|_0^1 \\
&= 1 + \frac{i}{3}
\end{aligned}$$

These examples show that in general the value of the integral depends not only on the endpoints of the integral but also the contour chosen.

On the other hand, if we consider

$$\int_0^{1+i} z dz$$

along the same three contours we have

(i)

$$\begin{aligned}
z &= (1 + i)t , dz = (1 + i)dt \\
\int_0^{1+i} z dz &= (1 + i)^2 \int_0^1 t dt = \frac{1}{2}(1 + i)^2 = i
\end{aligned}$$

(ii)

$$\begin{aligned}
z &= \begin{cases} x & 0 \leq x \leq 1 \\ 1 + iy & 0 \leq y \leq 1 \end{cases} \quad dz = \begin{cases} dx & 0 \leq x \leq 1 \\ idy & 0 \leq y \leq 1 \end{cases} \\
\int_0^{1+i} z dz &= \int_0^1 x dx + \int_0^1 (1 + iy) idy \\
&= \left. \frac{1}{2}x^2 \right|_0^1 + iy - \left. \frac{1}{2}y^2 \right|_0^1 \\
&= \frac{1}{2} + i - \frac{1}{2} = i
\end{aligned}$$

(iii)

$$\begin{aligned}
z &= t + it^2 , dz = (1 + 2it)dt \\
\int_0^{1+i} z dz &= \int_0^1 (t + it^2)(1 + 2it) dt \\
&= \int_0^1 (t + 3it^2 - 2t^3) dt = \left. \frac{1}{2}t^2 + it^3 - \frac{1}{2}t^4 \right|_0^1 \\
&= \frac{1}{2} + i - \frac{1}{2} = i
\end{aligned}$$

In these three cases we obtain the same value along each contour, and in addition this value is

$$\frac{1}{2}z^2 \Big|_0^{1+i}$$

Functions for which the value of the integral does not depend on the path are called ‘holomorphic’.

We have already seen that the function $f(z) = \bar{z}$ is not differentiable. On the other hand, the function $f(z) = z$ is entire.

This difference is reflected in the following result.

Cauchy’s Theorem.

If the function $f(z)$ is regular in a domain \mathcal{D} , and if $f'(z)$ is continuous, and if C is a scroc in \mathcal{D} whose interior contains only points of \mathcal{D} , then

$$\oint_C f(z) dz = 0$$

Equivalently, this theorem states that if $f(z)$ is regular in \mathcal{D} , then

$$\int_{z_0}^{z_1} f(z) dz$$

is independent of the choice of contour in \mathcal{D} , provided the contours enclose only points of \mathcal{D} .

To illustrate this last point, consider

$$\int_1^{-1} \frac{1}{z} dz$$

This function is regular except at $z = 0$.

If we take the contour $z = e^{i\theta}$, $0 \leq \theta \leq \pi$, we have

$$\begin{aligned} dz &= ie^{i\theta} d\theta \\ \int_1^{-1} \frac{1}{z} dz &= \int_0^\pi d\theta ie^{i\theta} e^{-i\theta} = \pi i \end{aligned}$$

while if we take the contour $z = e^{-i\theta}$ $0 \leq \theta \leq \pi$, we have

$$\begin{aligned} dz &= -ie^{-i\theta} d\theta \\ \int_1^{-1} \frac{1}{z} dz &= \int_0^\pi d\theta -ie^{-i\theta} e^{i\theta} = -\pi i \end{aligned}$$

Taken together these two contours enclose the singular point $z = 0$, so that Cauchy’s theorem does not apply.

Proof of Cauchy's Theorem

Suppose that $C_0(t)$, $0 \leq t \leq 1$ and $C_1(t)$, $0 \leq t \leq 1$ are two non-intersecting contours joining z_0 and z_1 in \mathcal{D} , whose interior is also in \mathcal{D} .

From these two contours we can construct a family $C_\alpha(t)$ of contours, $0 \leq \alpha \leq 1$, in the form

$$C_\alpha(t) = (1 - \alpha)C_0(t) + \alpha C_1(t)$$

which lie between C_0 and C_1 , and which coincide with these curves when $\alpha = 0, 1$ respectively.

For each value of α we can evaluate the integral to obtain

$$I(\alpha) = \int_0^1 f(z(t, \alpha)) \frac{\partial z(t, \alpha)}{\partial t} dt$$

Differentiating this expression we get

$$\begin{aligned} \frac{dI}{d\alpha} &= \int_0^1 f'(z) \frac{\partial z}{\partial \alpha} \frac{\partial z}{\partial t} dt \\ &\quad + \int_0^1 f(z) \frac{\partial^2 z}{\partial t \partial \alpha} dt \\ &= \int_0^1 f'(z) \frac{\partial z}{\partial \alpha} \frac{\partial z}{\partial t} dt \\ &\quad + f(z) \frac{\partial z}{\partial \alpha} \Big|_0^1 \\ &\quad - \int_0^1 f'(z) \frac{\partial z}{\partial t} \frac{\partial z}{\partial \alpha} dt \\ &= 0 \end{aligned}$$

since

$$\frac{\partial z}{\partial \alpha} = 0 \quad \text{when } t = 0, t = 1.$$

Hence $I(0) = I(1)$, which is what we wanted to prove.

Cauchy-Goursat Theorem.

The condition in Cauchy's Theorem that $f'(z)$ be continuous can be omitted, although the proof of the result becomes much more complicated.

In this form the result is known as the Cauchy-Goursat Theorem.

Multiply connected domains.

Suppose that C_1 and C_2 are scrocs with C_1 lying in the interior of C_2 , and that $f(z)$ is regular in the annular region bounded by C_1 and C_2 . Then

$$\oint_{C_1} f(z) dz = \oint_{C_2} f(z) dz .$$

Choose a point A on C_1 and a point B on C_2 and join them by a simple arc lying in the annulus.

Now consider the contour C consisting of

C_2 from B to B ;
 the arc BA ;
 $-C_1$ from A to A ;
 the arc AB ;

The function $f(z)$ is regular inside and on C , therefore

$$\oint_C f(z) dz = 0$$

$$\oint_{C_2} f(z) dz + \int_B^A f(z) dz + \oint_{-C_1} f(z) dz + \int_A^B f(z) dz = 0$$

$$\oint_{C_2} f(z) dz = \oint_{C_1} f(z) dz$$

This result extends to the general case:

Let \mathcal{D} be a multiply connected domain bounded externally by the scroc C and internally by the scrocs C_1, C_2, \dots, C_n . then if $f(z)$ is regular on \mathcal{D} ,

$$\oint_C f(z) dz = \sum_{r=1}^n \oint_{C_r} f(z) dz .$$

The Fundamental Theorem of the Calculus.

Suppose that $f(z)$ is continuous and holomorphic in the simply connected domain \mathcal{D} . Then there is a function $F(z)$, regular in \mathcal{D} , such that

$$F'(z) = f(z) \text{ in } \mathcal{D}$$

$$\int_{z_0}^{z_1} f(z) dz = F(z_1) - F(z_0) \quad z_0, z_1 \in \mathcal{D}$$

Proof:

Choose some fixed point c in \mathcal{D} .

Since $f(z)$ is holomorphic in \mathcal{D} ,

$$F(z) = \int_c^z f(\zeta) d\zeta$$

is uniquely defined for all contours in \mathcal{D} connecting c and z .

For any z_0 and z_1 in \mathcal{D} , we can evaluate the integral

$$\int_{z_0}^{z_1} f(z) dz$$

along a path passing through c .

Hence

$$\int_{z_0}^{z_1} f(z) dz = \int_{z_0}^c f(z) dz + \int_c^{z_1} f(z) dz = -F(z_0) + F(z_1)$$

To show that $F'(z) = f(z)$, consider

$$\begin{aligned}
 F(z+h) &= \int_c^{z+h} f(\zeta) d\zeta \\
 F(z) &= \int_c^z f(\zeta) d\zeta \\
 F(z+h) - F(z) &= \int_z^{z+h} f(\zeta) d\zeta \\
 f(z) &= \frac{1}{h} \int_z^{z+h} f(z) d\zeta \\
 \left| \frac{F(z+h) - F(z)}{h} - f(z) \right| &= \frac{1}{|h|} \left| \int_z^{z+h} (f(\zeta) - f(z)) d\zeta \right| \\
 &\leq \frac{1}{|h|} \max |f(\zeta) - f(z)| |h| \\
 &\leq \max |f(\zeta) - f(z)|
 \end{aligned}$$

However, f is continuous in \mathcal{D} , so that given any $\epsilon > 0$, we can find $\delta > 0$ such that

$$|f(\zeta) - f(z)| < \epsilon \quad \forall |\zeta - z| < \delta .$$

Therefore, if $|h| < \delta$,

$$\left| \frac{F(z+h) - F(z)}{h} - f(z) \right| < \epsilon$$

so that F is differentiable, and $F' = f$.

We have shown that a regular function is holomorphic, and that a holomorphic function is the derivative of a regular function.

To complete the process we will show that the derivative of a regular function is itself regular.

This behaviour distinguishes functions of a complex variable from real functions, for which the possession of one derivative is no guarantee of the existence of a second derivative.

In contrast, for complex functions, regularity implies the existence of derivatives of all orders.

This result will also justify our derivation of Laplace's equation from the Cauchy-Riemann equations.

A preliminary result:

$$\begin{aligned}
& \oint_{|z-z_0|=r} (z-z_0)^m dz \\
& z-z_0 = re^{i\theta} ; dz = ire^{i\theta} d\theta \\
& (z-z_0)^m = r^m e^{im\theta} \\
& \oint (z-z_0)^m dz = ir^{m+1} \int_0^{2\pi} e^{i(m+1)\theta} d\theta \\
& = ir^{m+1} \frac{1}{i(m+1)} e^{i(m+1)\theta} \Big|_0^{2\pi} \\
& = \frac{r^{m+1}}{m+1} (e^{i2(m+1)\pi} - 1) = 0 \\
& \quad \text{if } m \neq -1 \\
& = i \int_0^{2\pi} d\theta = 2\pi i \quad \text{if } m = -1
\end{aligned}$$

Cauchy's Integral Formula.

Suppose that $f(z)$ is regular inside and on the scroc C .

Then, if z_0 is in the interior of C ,

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-z_0} dz$$

Let C_δ be the circle, radius δ and centre z_0 , where δ is small enough to ensure that C_δ is inside C .

Then, the integrand

$$\frac{f(z)}{z-z_0}$$

is regular in the annular region between C_δ and C .

Therefore

$$\begin{aligned}
\oint_C \frac{f(z)}{z-z_0} dz &= \oint_{C_\delta} \frac{f(z)}{z-z_0} dz \\
&= \oint_{C_\delta} \frac{f(z_0)}{z-z_0} dz + \oint_{C_\delta} \frac{f(z)-f(z_0)}{z-z_0} dz \\
&= 2\pi i f(z_0) + \oint_{C_\delta} \frac{f(z)-f(z_0)}{z-z_0} dz
\end{aligned}$$

Since $f(z)$ is regular at z_0 ,

$$\lim_{z \rightarrow z_0} \frac{f(z)-f(z_0)}{z-z_0} = f'(z_0)$$

which means that, given any $\epsilon > 0$, we can find $\delta_1 > 0$ such that

$$\begin{aligned}
\left| \frac{f(z)-f(z_0)}{z-z_0} - f'(z_0) \right| &< \epsilon \quad \text{for all } |z-z_0| = \delta < \delta_1 \\
\left| \frac{f(z)-f(z_0)}{z-z_0} \right| &< |f'(z_0)| + \epsilon \\
\left| \oint_{C_\delta} \frac{f(z)-f(z_0)}{z-z_0} dz \right| &< (|f'(z_0)| + \epsilon) 2\pi\delta
\end{aligned}$$

which $\rightarrow 0$ as $\delta \rightarrow 0$.

Therefore

$$\frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} dz = f(z_0)$$

Consider

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z - \zeta} dz = F(\zeta)$$

where ζ is not on C .

Suppose that we consider differentiating with respect to ζ .

Since ζ is not on C , there is a positive constant k such that $|z - \zeta| \geq k$ for all $z \in C$, and we can choose $\delta\zeta$ so that $|\delta\zeta| < \frac{1}{2}k$.

Since

$$\frac{1}{z - \zeta}$$

is a regular function of ζ for $\zeta \neq z$, given $\epsilon > 0$ we can find $\delta(\epsilon, z)$ such that

$$\left| \frac{1}{\delta\zeta} \left(\frac{1}{z - (\zeta + \delta\zeta)} - \frac{1}{z - \zeta} \right) - \frac{1}{(z - \zeta)^2} \right| < \epsilon$$

for all $|\delta\zeta| < \delta(\epsilon, z)$.

If C has finite length l , we can find

$$\delta^* (> 0) = \min_{z \in C} \delta(\epsilon, z)$$

and if, in addition, $|f(z)| < M$ on C , then

$$\begin{aligned} & \left| \frac{F(\zeta + \delta\zeta) - F(\zeta)}{\delta\zeta} - \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - \zeta)^2} dz \right| \\ &= \left| \frac{1}{2\pi i} \int_C f(z) \left(\frac{1}{\delta\zeta} \left(\frac{1}{z - (\zeta + \delta\zeta)} - \frac{1}{z - \zeta} \right) - \frac{1}{(z - \zeta)^2} \right) dz \right| \\ &< \frac{1}{2\pi} M \epsilon l \quad \text{if } |\delta\zeta| < \delta^* \end{aligned}$$

Therefore the expression

$$\frac{1}{2\pi i} \int_C \frac{f(z)}{z - \zeta} dz$$

where C is a finite arc and f is bounded on C defines a function $F(\zeta)$ which is regular for all ζ not on C , and whose derivative is found by differentiating the integral with respect to ζ .

In fact, the procedure used above can be used to obtain derivatives of all orders:

$$F^{(n)}(\zeta) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z - \zeta)^{n+1}} dz$$

In particular, we can apply this result to Cauchy's Integral Formula.

If $f(z)$ is regular inside and on the scroc C , $f(z)$ has derivatives of all orders inside C , and

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz$$

In summary

If $f(z)$ is regular inside and on the *simple, closed, rectifiable, oriented curve* (scroc) C , then $f(z)$ has derivatives of all orders inside C , and for ζ inside C ,

$$\begin{aligned} f(\zeta) &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - \zeta)} dz \\ f'(\zeta) &= \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - \zeta)^2} dz \\ f^{(n)}(\zeta) &= \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - \zeta)^{n+1}} dz \end{aligned}$$

These results are known collectively as *Cauchy's Integral Formulae*.

Morera's Theorem.

We can now combine this result with the Fundamental theorem to obtain:

Let $f(z)$ be continuous on a domain \mathcal{D} , and suppose that

$$\oint_C f(z) dz = 0$$

for every scroc C contained in \mathcal{D} . The $f(z)$ is regular on \mathcal{D} .

This result is known as *Morera's Theorem*.

Cauchy's Inequalities.

If f is regular inside and on the circle $C_0 : |z - z_0| = R$, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \oint_{C_0} \frac{f(z)}{(z - z_0)^{n+1}} dz$$

It follows that

$$\begin{aligned} |f^{(n)}(z_0)| &\leq \frac{n!}{2\pi} \frac{M}{R^{n+1}} (2\pi R) \\ &\leq \frac{n!M}{R^n} \end{aligned}$$

where M is the maximum of $|f(z)|$ on C_0 .

These results are known as

Cauchy's Inequalities

Liouville's Theorem.

Suppose that $f(z)$ is *entire*; that is, it is regular throughout \mathbb{C} ; and that $|f(z)| \leq M$ for all z .

Then, for any point z_0 in \mathbb{C} , and any R , $f(z)$ is regular inside and on

$$C_0 ; |z - z_0| = R$$

and $|f(z)| \leq M$ on C_0 .

Therefore, using Cauchy's inequality

$$|f'(z_0)| \leq \frac{M}{R}$$

for all R . Therefore it follows that

$$|f'(z_0)| = 0$$

for all z_0 in \mathbb{C} .

i.e. $f(z)$ is a constant function.

This result

A bounded entire function is a constant

is known as *Liouville's Theorem*.

The Fundamental Theorem of Algebra.

A polynomial in \mathbb{C} of degree ≥ 1 , has a root in \mathbb{C} .

Suppose that

$$P(z) = a_0 + a_1z + \dots + a_mz^m$$

is a polynomial in \mathbb{C} such that the equation

$$P(z) = 0$$

has no solution for $z \in \mathbb{C}$.

Then $1/P(z)$ is an entire function of z which is bounded.

Therefore $1/P(z)$ is constant, and hence so is $P(z)$.

i.e. $P(z) \equiv a_0$, and P has degree 0.