Lucas' Theorem

If all the zeros of a polynomial $P(z)$ lie in a half plane, then all the zeros of the derivative $P'(z)$ lie in the same half plane.

Our first task is to characterise a half plane.

Suppose that the boundary of the half plane is the line

$$\zeta = a + tb$$

where a is a point on the line, $b \neq 0$ is the direction of the line, and t; $-\infty < t < \infty$, is a real parameter.

Consider a point z which is not on the line.

If we drop a perpendicular from z to the line, meeting it at $\zeta_0 = a + t_0 b$, then the vector $z - \zeta_0$ is perpendicular to the vector b.

This means

$$z - \zeta_0 = isb$$

where s is real, and $s > 0$ if z is to the left for t increasing, and $s < 0$ on the other side of the line.

Substituting for ζ_0 and simplifying we obtain

$$z - a = t_0 b + isb$$

$$\frac{z-a}{b} = t_0 + is$$

$$Im\left(\frac{z-a}{b}\right) = s$$

Therefore the two half planes determined by the line $z = a + tb$ are characterised by the conditions

$$Im\left(\frac{z-a}{b}\right) > 0 \quad \text{and} \quad Im\left(\frac{z-a}{b}\right) < 0\ .$$

Suppose that all the zeros $\alpha_1, \alpha_2, \ldots, \alpha_n$ of the polynomial

$$P(z) = c(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n)$$

lie in the half plane

$$Im\left(\frac{z-a}{b}\right) < 0\ .$$

This means that

$$Im\left(\frac{\alpha_i-a}{b}\right) < 0$$

for each α_i.

If z lies in the other half plane, then

$$Im\left(\frac{z-a}{b}\right) > 0$$

$$Im\left(\frac{z-\alpha_i}{b}\right) = Im\left(\frac{z-a}{b}\right) - Im\left(\frac{\alpha_i-a}{b}\right) > 0$$

$$Im\left(\frac{b}{z-\alpha_i}\right) < 0$$

Now

$$\frac{P'(z)}{P(z)} = \frac{1}{z-\alpha_1} + \cdots + \frac{1}{z-\alpha_n}$$

so that for any z in the half plane not containing the zeros of P,

$$Im\left(\frac{bP'(z)}{P(z)}\right) = \sum_{i=1}^{n} Im\left(\frac{b}{z-\alpha_i}\right) < 0\ ,$$

and consequently $P'(z) \neq 0$ in this half plane.

In a sharper formulation, this theorem tells us that the smallest convex polygon which contains the zeros of $P(z)$ also contains the zeros of $P'(z)$.