Polynomials

A polynomial is a function of the form

$$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z = a_0$$

where the coefficients a_i are complex numbers.

The number n is the **degree** of the polynomial, denoted $\partial(p)$.

A polynomial can be differentiated for all values of z - it is an **entire** function - and its derivative is the polynomial

$$p'(z) = n a_n z^{n-1} + (n-1) a_{n-1} z^{n-2} + \cdots + a_1$$

of degree $n - 1$.

Factorization

Definition. A polynomial $p(z)$ is *divisible* by $z - z_0$ if a polynomial $q(z)$ exists such that $p(z) = (z - z_0)q(z)$.

For any positive integer n, $z^n - z_0^n$ is divisible by $z - z_0$, since

$$\begin{aligned} z^n - z_0^n &= (z - z_0)(z^{n-1} + z^{n-2} z_0 + \cdots + z z_0^{n-2} + z_0^{n-1}) \\ &= (z - z_0) \sum_{r=0}^{n-1} z^{n-1-r} z_0^r \end{aligned}$$

From this it follows that if $p(z)$ is a polynomial, the polynomial $p(z) - p(z_0)$ is divisible by $z - z_0$.

$$\begin{aligned} \text{If} \quad p(z) &= \sum_{n=0}^{N} a_n z^n \\ p(z) - p(z_0) &= \sum_{n=1}^{N} a_n (z^n - z_0^n) \\ &= (z - z_0) \sum_{n=1}^{N} a_n \left(\sum_{r=0}^{n-1} z^{n-1-r} z_0^r \right) \\ &= (z - z_0) q(z) \end{aligned}$$

Hence $p(z)$ is divisible by $z - z_0$ if and only if $p(z_0) = 0$.

If $p(z_0) = 0$, then $p(z) = (z - z_0)q(z)$, while if $p(z) = (z - z_0)q(z)$, then $p(z_0) = (z_0 - z_0)q(z_0) = 0$.

The Fundamental Theorem of Algebra states that if $p(z)$ is a non-constant polynomial with complex coefficients, then the equation $p(z) = 0$ has a complex root. (This statement is not true if the words *complex* are replaced by *real* or *rational*.)

This theorem will be proved later in the course.

Assuming the theorem, we will prove that every complex polynomial of degree $n > 0$ can be factorized into linear factors in the form

$$p(z) = k(z - z_1)(z - z_2) \dots (z - z_n) \ .$$

Let $p_n(z)$ be a polynomial of degree n.

The Fundamental Theorem states that there is a complex number z_n such that $p_n(z_n) = 0$. Therefore, by our earlier result,

$$p_n(z) = (z - z_n)p_{n-1}(z) \ ,$$

where $p_{n-1}(z)$ is a polynomial of degree $n - 1$ in z.

By the Fundamental Theorem, this new polynomial also has a complex root; z_{n-1} say; so that

$$p_{n-1}(z) = (z - z_{n-1})p_{n-2}(z) \ .$$

Proceeding recursively, we find

$$p_n(z) = (z - z_n)(z - z_{n-1}) \dots (z - z_1)p_0(z)$$

where $p_0(z)$ is a polynomial of degree 0; that is, a constant.

This shows that all the roots of the polynomial are in $\mathbb{C}$.

Complex numbers themselves arose from a need to solve real quadratic equations. This result shows that there is no need to invent a new class of numbers to solve polynomial equations (of any order) with complex coefficients. Since real numbers can be considered as a special case of complex numbers, this also means that complex numbers suffice for finding all the roots of any real polynomial equation.

We say that $\mathbb{C}$ is *algebraicly complete*.

Logarithmic derivatives

Given a regular function f, the ratio

$$\frac{f'}{f}$$

is called the **logarithmic derivative** of f.

Where f is a polynomial, this expression has a simple partial fraction expansion.

$$\text{If} \quad p(z) = k\prod_{i=1}^{n}(z - z_i) \ , \ \frac{p'(z)}{p(z)} = \sum_{i=1}^{n} \frac{1}{z - z_i} \ .$$

The result is trivially true when $n = 1$.

Assume that it is true for $q(z) = k(z - z_1)(z - z_2) \dots (z - z_{n-1})$, and consider $p(z) = q(z)(z - z_n)$.

$$\begin{aligned} p'(z) &= q'(z)(z - z_n) + q(z).1 \\ \frac{p'(z)}{p(z)} &= \frac{q'(z)}{q(z)} + \frac{1}{z - z_n} \\ &= \sum_{r=1}^{n-1} \frac{1}{z = z_r} + \frac{1}{z - z_n} = \sum_{r=1}^{n} \frac{1}{z - z_r} \end{aligned}$$

Therefore, by the principle of mathematical induction, the result holds for all integers $n \geq 1$.