

DIFFERENTIATION

The derivative of a function of a complex variable is defined in the same way as that of a function of a real variable.

If $f(z)$ is defined in a neighbourhood of the point z_0 , then f is differentiable at z_0 if

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists, and the value of this limit is the derivative of $f(z)$ at z_0 , denoted $f'(z_0)$.

Note that if the derivative exists at z_0 , then

$$\begin{aligned} \lim_{z \rightarrow z_0} (f(z) - f(z_0)) &= \lim_{z \rightarrow z_0} \left(\frac{f(z) - f(z_0)}{z - z_0} \right) (z - z_0) \\ &= \left(\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0} \right) \left(\lim_{z \rightarrow z_0} (z - z_0) \right) \\ &= f'(z_0) \cdot 0 = 0 \end{aligned}$$

so that $f(z)$ is continuous at z_0 .

For example, if $f(z) = z^2$, the derivative exists for all $z_0 \in \mathbb{C}$, and

$$\begin{aligned} f'(z_0) &= \lim_{z \rightarrow z_0} \frac{z^2 - z_0^2}{z - z_0} \\ &= \lim_{z \rightarrow z_0} (z + z_0) = 2z_0 \end{aligned}$$

Where the derivative exists at all points in some region, we call the function defined by these values the *derived function* or *derivative* of the original function, and denote it by $f'(z)$.

The above example shows that if $f(z) = z^2$, $f'(z) = 2z$.

A similar construction can be used to show that if $f(z) = z^n$, $f'(z) = nz^{n-1}$.

A function (like z^2) which is differentiable at every point of \mathbb{C} is said to be **entire**.

Of the elementary functions, polynomials are entire.

On the other hand, $f(z) = 1/z$ is not entire since it is not differentiable at $z = 0$. (It is not even defined there!)

The function $f(z) = zz^*$ is differentiable at $z = 0$, and $f'(0) = 0$.

$$\begin{aligned} f'(0) &= \lim_{z \rightarrow 0} \frac{zz^* - 0}{z - 0} \\ &= \lim_{z \rightarrow 0} z^* = 0 \end{aligned}$$

In this case (as we shall shortly prove) this is the only point at which this function is differentiable.

Since the definition of derivative has the same form as for the real case, the elementary results concerning the derivatives of sums and products of functions carry over.

For example: If $f(z)$ and $g(z)$ are differentiable at z_0 , so are

(i) $f(z) \pm g(z)$, and $(f \pm g)'(z_0) = f'(z_0) \pm g'(z_0)$;

- (ii) $f(z).g(z)$, and $(fg)'(z_0) = f'(z_0)g(z_0) + f(z_0)g'(z_0)$;
 (iii) $f(z)/g(z)$ provided $g(z_0) \neq 0$, and

$$\left(\frac{f}{g}\right)'(z_0) = \frac{f'(z_0)g(z_0) - f(z_0)g'(z_0)}{g^2(z_0)}$$

The proof of the result (ii) for example is;

$$\begin{aligned} (fg)'(z_0) &= \lim_{z \rightarrow z_0} \frac{f(z)g(z) - f(z_0)g(z_0)}{z - z_0} \\ &= \lim_{z \rightarrow z_0} \frac{f(z)g(z) - f(z)g(z_0) + f(z)g(z_0) - f(z_0)g(z_0)}{z - z_0} \\ &= \lim_{z \rightarrow z_0} \frac{f(z)(g(z) - g(z_0))}{z - z_0} + \lim_{z \rightarrow z_0} \frac{(f(z) - f(z_0))g(z_0)}{z - z_0} \\ &= \left(\lim_{z \rightarrow z_0} f(z)\right) \left(\lim_{z \rightarrow z_0} \frac{g(z) - g(z_0)}{z - z_0}\right) \\ &\quad + \left(\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}\right) \left(\lim_{z \rightarrow z_0} g(z_0)\right) \\ &= f(z_0)g'(z_0) + f'(z_0)g(z_0) \end{aligned}$$

We also have the result for compound functions (the Chain rule):

If g is differentiable at z_0 , and f is differentiable at $g(z_0)$, then $f(g(z))$ is differentiable at z_0 , and

$$(f(g))'(z_0) = f'(g(z_0))g'(z_0)$$

For example, we can write z^4 as $f(f(z))$, where $f(z) = z^2$, so that the derivative at z_0 is

$$f'(f(z_0))f'(z_0) = (2z_0^2)(2z_0) = 4z_0^3$$

THE CAUCHY-RIEMANN EQUATIONS

If the function $f(z) = u(x, y) + iv(x, y)$ is differentiable at the point $z_0 = x_0 + iy_0$, then the limit

$$\lim_{z \rightarrow z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

exists irrespective of the manner in which z approaches z_0 .

In particular, the limit exists if we let z approach z_0 along the line $y = y_0$.

In this case $z - z_0 = x - x_0$, and

$$\begin{aligned} &f'(z_0) \\ &= \lim_{x \rightarrow x_0} \frac{(u(x, y_0) + iv(x, y_0)) - (u(x_0, y_0) + iv(x_0, y_0))}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \frac{u(x, y_0) - u(x_0, y_0)}{x - x_0} \\ &\quad + i \lim_{x \rightarrow x_0} \frac{v(x, y_0) - v(x_0, y_0)}{x - x_0} \\ &= u_x(x_0, y_0) + i v_x(x_0, y_0) \end{aligned}$$

On the other hand, the same limit exists if we let z approach z_0 along the line $x = x_0$.

In this case $z - z_0 = i(y - y_0)$, and

$$\begin{aligned} & f'(z_0) \\ &= \lim_{y \rightarrow y_0} \frac{(u(x_0, y) + iv(x_0, y)) - (u(x_0, y_0) + iv(x_0, y_0))}{i(y - y_0)} \\ &= -i \lim_{y \rightarrow y_0} \frac{u(x_0, y) - u(x_0, y_0)}{y - y_0} \\ &\quad + \lim_{y \rightarrow y_0} \frac{v(x_0, y) - v(x_0, y_0)}{y - y_0} \\ &= -iu_y(x_0, y_0) + v_y(x_0, y_0) \end{aligned}$$

Comparing these results, we see that, if the derivative exists at z_0 ,

$$\begin{aligned} u_x(x_0, y_0) &= v_y(x_0, y_0) \\ u_y(x_0, y_0) &= -v_x(x_0, y_0) \end{aligned}$$

These two equations are known as the Cauchy-Riemann equations.

e.g. We have seen that $|z|^2 = zz^*$ is differentiable at $z = 0$.

In this case, $u = x^2 + y^2$ and $v = 0$.

We have $u_x = 2x$, $u_y = 2y$, $v_x = 0$ and $v_y = 0$, so that when $z = 0$,

$$\begin{aligned} u_x = 0 &= v_y \\ u_y = 0 &= -v_x \end{aligned}$$

as expected.

Since there are no other points in the complex plane at which the Cauchy-Riemann equations are satisfied, the function $|z|^2$ is differentiable at only one point.

It should be noted that satisfying the Cauchy-Riemann equations alone does not guarantee the existence of the derivative.

The converse theorem requires that the partial derivatives be continuous at the point (x_0, y_0) . However, this requirement is met by the functions which appear in this course.

e.g. Consider $f(z) = e^z$.

$$\begin{aligned} u(x, y) &= e^x \cos y \\ v(x, y) &= e^x \sin y \end{aligned}$$

$$\begin{aligned} u_x &= e^x \cos y \\ v_y &= e^x \cos y \end{aligned}$$

$$\begin{aligned} u_y &= -e^x \sin y \\ v_x &= e^x \sin y \end{aligned}$$

Since e^x is continuous for all x and $\sin y$ and $\cos y$ are continuous for all y , and since $u_x = v_y$ and $u_y = -v_x$ for all x and y , the function e^z is differentiable for all z . (It is entire.)

Note that the derivative $f'(z_0)$ can be calculated as either

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

or

$$f'(z_0) = v_y(x_0, y_0) - iu_y(x_0, y_0)$$

e.g.

$$\begin{aligned} \frac{d}{dz}e^z &= u_x + iv_x \\ &= e^x \cos y + ie^x \sin y = e^z \end{aligned}$$

Functions like $|z|^2$ which are differentiable only at isolated points are of no further interest.

We concentrate on functions which are differentiable at every point of some open connected set of points in \mathbb{C} . Such a set is called a domain, denoted \mathcal{D} .

Such a function is said to be 'regular' on the domain \mathcal{D} .

e.g.

Polynomials in z are regular on \mathbb{C} .

e^z is regular on \mathbb{C} .

$1/z$ is regular on $\mathbb{C} \setminus \{0\}$.

$\sum_{n=1}^{\infty} z^{n!}$ is regular on $|z| < 1$.

Suppose that $f(z)$ is regular on \mathcal{D} , and that the functions u and v have continuous second order partial derivatives. (We will see later that the second condition is in fact implied by the first.)

Then, from the Cauchy-Riemann equations

$$\begin{aligned} u_{xx} &= v_{xy} \\ u_{yy} &= -v_{yx} \\ \text{but } v_{xy} &= v_{yx} \quad (\text{continuity}) \\ \text{therefore } u_{xx} + u_{yy} &= 0 \end{aligned}$$

Similarly

$$v_{xx} + v_{yy} = 0$$

The real and imaginary parts of a function regular on \mathcal{D} are solutions of Laplace's equation on \mathcal{D} . The functions u and v are said to be harmonic.

Since they satisfy the Cauchy-Riemann equations, u and v are related. If we know u , then we can determine v to within a constant.

e.g. Consider $u(x, y) = x^3 - 3xy^2$.

$$\begin{aligned}u_x &= 3x^2 - 3y^2 \\u_{xx} &= 6x \\u_y &= -6xy \\u_{yy} &= -6x \\u_{xx} + u_{yy} &= 0\end{aligned}$$

Therefore u is harmonic.

If $u + iv$ is a regular function,

$$\begin{aligned}v_y &= u_x = 3x^2 - 3y^2 \\v &= 3x^2y - y^3 + c(x) \\v_x &= 6xy + c'(x) \\ \text{but } v_x &= -u_y = 6xy \\ \text{therefore } c'(x) &= 0 ; c \text{ is constant}\end{aligned}$$

In fact, $u + iv = z^3 + ic$.

EXERCISES

1. Write $f(z) = u(x, y) + iv(x, y)$ and find $u(x, y), v(x, y)$ in each of the following cases:

- (i) $z^2 + 2z$
- (ii) $\frac{1}{z} \quad (z \neq 0)$
- (iii) $\sinh z$
- (iv) $\arg z \quad (0 < \arg z < \pi)$

2. Differentiate the functions (i), (ii) and (iii) from question 1.

3. Use the Cauchy-Riemann Equations to determine the points at which the function (iv) question 1 is differentiable.

4. Show that u is harmonic in some domain, and find a harmonic conjugate v , when

- (a) $u = 2x(1 - y)$
- (b) $u = 2x - y^3 + 3x^2y$
- (c) $u = \sinh x \sin y$
- (d) $u = \frac{y}{x^2 + y^2}$