FUNCTIONS

A **function** is a rule which assigns to every number z in some subset $\mathcal{D}$ of $\mathbb{C}$ a unique number w in some other subset $\mathcal{R}$ of $\mathbb{C}$. Genericly we write $w = f(z)$.

The set $\mathcal{D}$ is called the **domain of definition** of the function. If it is not explicitly defined it is assumed to be the largest set for which the definition of the function is valid.

The set $\mathcal{R}$ of values assumed by the function is called the **range** of the function.

For example:

If $f(z) = z + 2$, then this definition is valid for all $z \in \mathbb{C}$, so that the domain of definition is $\mathbb{C}$.

The range of this function is also $\mathbb{C}$.

If $f(z) = 1/z$, then the domain is $\mathbb{C} \setminus \{0\}$, as is the range.

If $f(z) = |z|$, then the domain is $\mathbb{C}$ and the range is the set

$$\{Im(w) = 0, Re(w) \geq 0\}$$

Sometimes the way we look at a function may affect the answers.

The function $f_1(z)$ given by the infinite geometric series

$$f_1(z) = \sum_{n=0}^{\infty} z^n$$

is defined for the points at which the series converges; namely $|z| < 1$. Its range is the set of complex numbers w with $Re(w) > \frac{1}{2}$.

On the other hand, the function

$$f_2(z) = \frac{1}{1-z}$$

is defined for all $z \in \mathbb{C}$ except $z = 1$, and its range is the set of all complex numbers except $w = 0$.

These two functions take precisely the same values for $|z| < 1$. In cases like this it is usual to consider f_2 as the function, and f_1 as a local representation.

In fact, f_1 is the **Taylor series expansion** of f_2 about $z = 0$.

Exercises.

1. Determine the domain of definition of the following functions:

(i) $$f(z) = \frac{1}{|z| - 3}$$

(ii) $$f(z) = \frac{3z}{(z-1)(z-2)}$$

(iii) $$f(z) = \sum_{n=1}^{\infty} z^{n!}$$

2. A function f with domain $\mathbb{C}$ is said to be *odd* if $f(z) = -f(-z)$ for all z, and *even* if $f(z) = f(-z)$ for all z.

Classify the following functions as odd, even, or neither odd nor even.

(i) $f(z) = |z|$
(ii) $f(z) = Re(z)$
(iii) $f(z) = Im(z)$
(iv) $f(z) = z^*$

Real and Imaginary parts.

Writing $z = x + iy$ and $w = u + iv$, we can represent a function in the form

$$f(z) = u(x, y) + iv(x, y)$$

where $u(x, y)$ and $v(x, y)$ are real functions of two real variables.

For example, if $f(z) = z^3$, then we have

$$\begin{aligned}(x + iy)^3 &= x^3 + 3x^2(iy) + 3x(iy)^2 + (iy)^3 \\ &= x^3 + i3x^2y - 3xy^2 - iy^3 \\ u(x, y) &= x^3 - 3xy^2 \\ v(x, y) &= 3x^2y - y^3\end{aligned}$$

This gives an alternative representation of the function as a mapping of points in $\mathbb{R}^2$ onto points in $\mathbb{R}^2$.

We use this representation to provide a pictorial representation of f.

We choose a grid in the $x - y$ plane and plot its image in the $u - v$ plane (or vice-versa).

For example, consider $f(z) = 1/z$.

$$\begin{aligned}\frac{1}{z} &= \frac{z^*}{zz^*} = \frac{x - iy}{x^2 + y^2} \\ u(x, y) &= \frac{x}{x^2 + y^2} \\ v(x, y) &= -\frac{y}{x^2 + y^2}\end{aligned}$$

If we consider a rectangular grid in the $x-y$ plane, then the lines parallel to the x axis are represented parametricly by

$$x = t\ ;\ -\infty < t < \infty\ ; y = c\ .$$

The image of such a line in the $u - v$ plane is given by

$$\begin{aligned}u &= \frac{t}{t^2 + c^2} \\ v &= -\frac{c}{t^2 + c^2} \\ u^2 + v^2 &= \frac{t^2 + c^2}{(t^2 + c^2)^2} = \frac{1}{t^2 + c^2} \\ v &= -c(u^2 + v^2)\end{aligned}$$

When $c \neq 0$, this represents a circle centered at $(0, -1/(2c))$ and passing through the origin $(u, v) = (0, 0)$. This common point on all the circles is the image of the point at infinity.

When $c = 0$, i.e. when the original line is the x axis, its image is the u axis in the $u - v$ plane.

Similarly, for lines parallel to the y axis we consider

$$x = c \; ; \; y = t \; ; \; -\infty < y < \infty \; .$$

In this case the images are given by

$$\begin{aligned} u &= \frac{c}{t^2 + c^2} \\ v &= -\frac{t}{c^2 + t^2} \\ u &= c(u^2 + v^2) \end{aligned}$$

Again, when $c \neq 0$, this represents a circle passing through the origin. In this case the centre is the point $(1/(2c), 0)$.

Equally, the y axis maps onto the v axis in the $u - v$ plane.

In this case, the mapping is $1-1$; each point in the range is the image of precisely one point in the domain of definition.

Now consider the function $f(z) = z^2$.

$$\begin{aligned} (x + iy)^2 = x^2 + 2x(iy) + (iy)^2 &= x^2 + i2xy - y^2 \\ u(x, y) &= x^2 - y^2 \\ v(x, y) &= 2xy \end{aligned}$$

For lines parallel to the x axis; $x = t$, $y = c$, we have

$$u = t^2 - c^2 \; ; \; v = 2tc$$

If $c \neq 0$,

$$t = \frac{v}{2c} \; ; \; u = \frac{1}{4c^2}v^2 - c^2$$

which represents a parabola with its axis along the u axis and its vertex at $u = -c^2$.

Note that the lines $y = c$ and $y = -c$ in the $x - y$ plane are mapped onto precisely the same parabola.

When $c = 0$, we have $v = 0$ and $u = t^2 \geq 0$, so that the x axis is mapped onto the positive u axis.

The lines parallel to the y axis; $x = c$, $y = t$; map onto

$$u = c^2 - t^2 \; ; \; v = 2ct$$

When $c = 0$, this gives the negative real axis in the w plane.

Otherwise, we have as before

$$t = \frac{v}{2c} \; ; \; u = c^2 - \frac{1}{4c^2}v^2$$

which represents a parabola pointing in the opposite direction to the previous case. Again, $x = c$ and $x = -c$ are mapped onto the same parabola.

Consider the lines $x = 1$ and $y = 1$. These lines have a unique intersection at $z = 1 + i$ in the z plane.

However, their images

$$u = 1 - \frac{1}{4}v^2$$

and

$$u = \frac{1}{4}v^2 - 1$$

intersect in the w plane at two places;

$u = 0, v = 2$ which corresponds to the common point $z = 1 + i$,

and

$u = 0, v = -2$ which corresponds to the point $z = 1 - i$ on the line $x = 1$ but to the point $z = -1 + i$ on the line $y = 1$.

We can overcome this anomaly by using two copies of the w plane, so that, for example, the lines $x = c$ and $x = -c$ are mapped on different planes. If we do this, the y axis, $x = 0$, maps onto the negative real axis of both planes.

If we consider the line $y = 1$, the image of the section with $x > 0$ appears on the first plane (and intersects the image of $x = 1$ there), while the image of the section $x < 0$ is on the second plane. We obtain a continuous image if we (mentally) join the first plane to the second along the negative real axis so that the upper part of the first plane merges smoothly into the lower part of the second.

Considering the line $y = -1$ in a similar fashion, we see that we also need to merge upper part of the second plane with the lower part of the first plane along the same line. (To achieve this in practice we need a four dimensional space in which to work.)

The surface which results from this procedure is called a **Riemann surface**.

The line along which we merge the planes is called a **branch cut**, and the ends of this line, $w = 0$ and $w = \infty$ are called **branch points.**

The branch points are fixed by the function, but the branch cut can be chosen arbitrarily between these endpoints.

The function $w = z^2$ is now represented as a $1 - 1$ mapping from the z plane onto this Riemann surface.

Since it is $1 - 1$, we can invert it.

If we choose the branch cut given above, this inversion gives us two functions $z = w^{1/2}$ defined on a w plane which is cut from ∞ to 0 along the negative real axis.

For the first, $Re(w^{1/2}) > 0$, and for the second $Re(w^{1/2}) < 0$.

Exercises.

3. Express the following functions in the form $u(x, y) + iv(x, y)$;

(i) $$f(z) = \frac{1}{1 + z}$$

(ii) $$f(z) = z^4$$

(iii) $$f(z) = (z + 1)(z^* - 1)$$

4. Determine the image in the w plane of the set of points $\{|z| \leq \frac{1}{2}\}$ under the mapping $w = 1/(z + 1)$.

5. Determine the images of the lines $y = x+1$ and $y = 2x-1$ under the mapping $w = z^2$.

The exponential function

We have seen the representation

$$\cos\theta + i\sin\theta = e^{i\theta} \ .$$

We use this to define the exponential function for an arbitrary complex number:

$$\begin{aligned}\exp(z) = e^z &= e^{x+iy} = e^x e^{iy} \\ &= e^x\cos y + ie^x \sin y\end{aligned}$$

(Remember when evaluating this function that y has radian measure.)

From the definition we have immediately the following properties;

(i) $e^0 = 1$.
(ii) $|e^z| = e^x$.
(iii) e^z is not zero for any z.
(iv) $1/e^z = e^{-z}$.
(v) $(e^z)^n = e^{nz}$.
(vi) $e^{\pi i} = -1$ and $e^{2\pi i} = 1$.
Hence, $e^{z+2\pi i} = e^z$;
the exponential function is periodic
with period $2\pi i$.

Graphical representation of the exponential function

If we consider the domain $\mathcal{D}$ consisting of the infinite strip

$$-\pi < \mathrm{Im} z < \pi$$

then the exponential function maps this domain one-to-one onto the complex plane with the exception of the negative real axis.

The strip is called a *fundamental strip* for the exponential function, and the region onto which it is mapped is called a *cut plane.*

Points just above the cut correspond to

$$\exp(a + i(\pi-))$$

while points below the cut correspond to

$$\exp(a + i(-\pi+))$$

The cut is also called a *branch cut*, and the ends of the cut ($w = 0$ and $w = \infty$) are called *branch points.*

Choosing a different fundamental strip,

$$b < \mathrm{Im} z < b + 2\pi$$

produces a different branch cut in general, but does not change the branch points.

If we choose the strip

$$\pi < \mathrm{Im} z < 3\pi$$

then the cut again lies along the negative real axis. In this case, points just below the cut correspond to

$$\exp(a + i(\pi+))$$

and can be considered as the continuation of of the points above the cut in the first case.

If we join these two cut planes appropriately across the cut, we have a *spiral staircase* like structure which represents the map of the double strip

$$-\pi < \mathrm{Im} z < 3\pi$$

and we could continue in like fashion to add cut planes top and bottom to create an infinite spiral which is the map of $\mathbb{C}$ under the exponential function.

This structure is another example of a *Riemann surface*.

Trigonometric functions

We have seen the real trigonometric functions defined in terms of $e^{i\theta}$.

We extend this form to define the sin and cos functions for complex values of the arguments.

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}$$
$$\cos z = \frac{e^{iz} + e^{-iz}}{2}$$

These functions are defined for all $z \in \mathbb{C}$.

Note that these definitions are similar to those of the hyperbolic functions.

$$\sinh z = \frac{e^{z} - e^{-z}}{2}$$
$$\cosh z = \frac{e^{z} + e^{-z}}{2}$$

Indeed, these functions are related by the formulae

$$\cos(iz) = \cosh z$$
$$\cosh(iz) = \cos z$$
$$\sin(iz) = i \sinh z$$
$$\sinh(iz) = i \sin z$$

Using the definition of the exponential function, we have

$$\begin{aligned}
e^{iz} &= e^{ix-y} = e^{-y}(\cos x + i\sin x) \\
e^{-iz} &= e^{-ix+y} = e^{y}(\cos x - i\sin x) \\
e^{iz} - e^{-iz} &= \cos x(e^{-y} - e^{y}) + i\sin x(e^{-y} + e^{y}) \\
&= -2\cos x \sinh y + 2i \sin x \cosh y \\
\sin z &= \sin x \cosh y + i \cos x \cosh y
\end{aligned}$$

$$\begin{aligned}
e^{iz} + e^{-iz} &= \cos x(e^{-y} + e^{y}) + i\sin x(e^{-y} - e^{y}) \\
&= 2\cos x \cosh y - 2i \sin x \sinh y \\
\cos z &= \cos x \cosh y - i \sin x \sinh y
\end{aligned}$$

From these results we can calculate

$$\begin{aligned}
|\sin z|^2 &= \sin^2 x \cosh^2 y + \cos^2 x \sinh^2 y \\
&= \sin^2 x(1 + \sinh^2 y) + \cos^2 x \sinh^2 y \\
&= \sin^2 x + (\sin^2 x + \cos^2 x)\sinh^2 y \\
&= \sin^2 x + \sinh^2 y
\end{aligned}$$

so that if $\sin z = 0$, then $\sin x = 0$ and $\sinh y = 0$.

Since the last equation is only satisfied by $y = 0$, we see that the only zeros of $\sin z$ are the real zeros of the real function $\sin x$.

Similarly,

$$|\cos z|^2 = \cos^2 x + \sinh^2 y$$

and the only zeros of $\cos z$ are the real zeros of $\cos x$.

Exercises.

6. Find all the values of z for which $e^z = -1 + i\sqrt{3}$.
7. Find all the values of z for which $\sin z = 2$.
8. Determine the images in the w plane of the grid

$$\begin{aligned}
x &= \frac{k\pi}{4}; -2 \le k \le 2; y \ge 0 \\
y &= 0, 1, 2; -\frac{\pi}{2} \le x \le \frac{\pi}{2}
\end{aligned}$$

under the mapping $w = \sin z$.

Limits and Continuity.

Suppose that the function $f(z)$ is defined in a neighbourhood of the point $z = a$ (though not necessarily at a).

We say that $f(z)$ has a limit l at a, written

$$\lim_{z \to a} f(z) = l$$

if, given any $\epsilon > 0$, there is a $\delta > 0$ such that

$$|f(z) - l| < \epsilon \text{ for all } 0 < |z - a| < \delta\ .$$

In this definition ϵ and δ are real numbers.

For example:
If $f(z) = c$ for all z (i.e. f is a constant function) then, for any $a \in \mathbb{C}$

$$\lim_{z \to a} f(z) = c \ .$$

Given any $\epsilon > 0$ we can choose any $\delta > 0$, and

$$|f(z) - l| = |c - c| = 0 < \epsilon \ \forall \ 0 < |z - a| < \delta \ .$$

If $f(z) = z$, then, for any $a \in \mathbb{C}$,

$$\lim_{z \to a} f(z) = a \ .$$

Given any $\epsilon > 0$, choose $\delta = \epsilon$. Then

$$|f(z) - l| = |z - a| < \epsilon \ \forall \ 0 < |z - a| < \delta \ .$$

$$\lim_{z \to i} \frac{z^3 + i}{z - i} = -3 \ .$$

In this case the function is not defined at $z = i$.
However, for $z \neq i$, we have

$$\frac{z^3 + i}{z - i} = z^2 + iz - 1$$

so that

$$\frac{z^3 + i}{z - i} - (-3) = z^2 + iz + 2 = (z - i)(z + 2i)$$

$$\left| \frac{z^3 + i}{z - i} - (-3) \right| = |z - i||z + 2i|$$

$$|z + 2i| = |z - i + 3i| \leq |z - i| + |3i| = |z - i| + 3$$

If $|z - i| < 1$, $|z + 2i| < 4$, so that

$$|f(z) + 3| < 4|z - i| \quad \text{for} \quad 0 < |z - i| < 1$$

Hence, given any $\epsilon > 0$,

$$|f(z) + 3| < \epsilon \quad \text{for} \quad 0 < |z - i| < \min(1, \epsilon/4) \ .$$

The definition has the same structure as the definition of limit for real functions, so that we can once again carry over the elementary theorems about limits of real functions.

In particular, provided $\lim_{z \to a} f(z)$ and $\lim_{z \to a} g(z)$ exist, we have

(i) $$\lim_{z \to a} (f(z) + g(z)) = \lim_{z \to a} f(z) + \lim_{z \to a} g(z)$$

(ii) $$\lim_{z \to a} cf(z) = c \lim_{z \to a} f(z)$$

(iii) $$\lim_{z \to a} f(z)g(z) = \left(\lim_{z \to a} f(z) \right) \left(\lim_{z \to a} g(z) \right)$$

and, provided $\lim_{z\to a} g(z) \neq 0$,

(iv) $$\lim_{z\to a} \frac{f(z)}{g(z)} = \frac{\lim_{z\to a} f(z)}{\lim_{z\to a} g(z)}$$

We also have the useful result;
If, for some δ, $f(z) = g(z)$ for $0 < |z-a| < \delta$, then

$$\lim_{z\to a} f(z) = \lim_{z\to a} g(z) \ .$$

These results can be used in an alternative proof of the previous example.
For $z \neq i$, $(z^3+i)/(z-i) = z^2 + iz - 1$, so that

$$\begin{aligned}
\lim_{z\to i} \frac{z^3+i}{z-i} &= \lim_{z\to i}(z^2+iz-1) \\
&= \lim_{z\to i} z^2 + \lim_{z\to i} iz - \lim_{z\to i} 1 \\
&= \left(\lim_{z\to i} z\right)^2 + i \lim_{z\to i} z - 1 \\
&= i^2 + ii - 1 = -1 - 1 - 1 = -3
\end{aligned}$$

We have seen that the constant function $f(z) = c$ and the identity function $f(z) = z$ have the useful property that

$$\lim_{z\to a} f(z) = f(a) \ ,$$

and it is possible to generalise the derivation above to show that this also holds for any polynomial function

$$f(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0 \ .$$

We have shown that it is true if f is a polynomial of degree 0, that is a constant.

Suppose that it is true for any polynomial of degree $n-1$ and consider the polynomial function

$$f(z) = c_n z^n + c_{n-1} z^{n-1} + \cdots + c_1 z + c_0 \ .$$

We can write this function as

$$\begin{aligned} f(z) &= z(c_n z^{n-1} + c_{n-1} z^{n-1} + \cdots + c_1) + c_0 \\ &= z g(z) + c_0 \end{aligned}$$

where $g(z)$ is a polynomial of degree $n-1$.

Then

$$\lim_{z \to a} f(z) = \left(\lim_{z \to a} z\right) \left(\lim_{z \to a} g(z)\right) + \lim_{z \to a} c_0 = a g(a) + c_0 = f(a) \ .$$

Therefore, by the principle of Mathematical Induction, the result holds for polynomial functions of all orders.

This property is called **continuity**.

Specifically, a function $f(z)$ is *continuous* at $z = a$, if

(i) $f(z)$ is defined at $z = a$;

(ii) $\lim_{z \to a} f(z)$ exists;

(iii) $\lim_{z \to a} f(z) = f(a)$.

.

In terms of ϵ and δ, the function f is continuous at a if, given any $\epsilon > 0$, we can find $\delta > 0$ such that

$$|f(z) - f(a)| < \epsilon \quad \text{for all} \quad |z - a| < \delta \ .$$

Note how this definition implicitly assumes the existence of $f(a)$.

Note also that it is not necessary to restrict z away from a since the inequality is automaticly satisfied there.

The function $(z^3 + i)/(z - i)$ is not continuous at $z = i$ although the limit exists, because it is not defined at that point.

On the other hand, the function

$$f(z) = \begin{cases} \frac{z^3+i}{z-i} & z \neq i \\ -3 & z = i \end{cases}$$

is continuous at $z = i$ since $f(i)$ is now defined and has the same value as the limit.

The results for limits stated above carry over to continuity.

Let f and g be continuous at a. Then:

(i) $f+g$ is continuous at a;

(ii) $f.g$ is continuous at a;

(iii) f/g is continuous at a provided $g(a) \neq 0$.

We also have the compound result:

If $g(z)$ is continuous at a, and $f(z)$ is continuous at $g(a)$, then $f(g(z))$ is continuous at a.

Since f is continuous at $g(a)$, given any $\epsilon > 0$ we can find a $\delta_1 > 0$ such that

$$|f(w) - f(g(a))| < \epsilon \quad \text{for all} \quad |w - g(a)| < \delta_1 \ .$$

Since g is continuous at a, for this value $\delta_1 > 0$ we can find a $\delta > 0$ such that

$$|g(z) - g(a)| < \delta_1 \quad \text{for all} \quad |z - a| < \delta \ .$$

But if $|g(z) - g(a)| < \delta_1$, $|f(g(z)) - f(g(a))| < \epsilon$, so that

$$|f(g(z)) - f(g(a))| < \epsilon \quad \text{for all} \quad |z - a| < \delta \ .$$

The definition of continuity applies at individual points.

While it is possible to construct pathalogical functions which are continuous at isolated points, the more common occurrence is that the function f will be continuous at every points of some set S.

We then say that f is continuous on S.

For example, polynomial functions are continuous on $\mathbb{C}$, while the function $(z^3 + i)/(z - i)$ is continuous on $\mathbb{C} \setminus \{i\}$.

Exercises.

9. Use the ϵ, δ definition to show that

$$\lim_{z \to -i} \frac{z^3 - iz^2 - 2i}{z + i} = -5 \ .$$

10. Prove that if f and g are continuous at a, so is $f + g$.