The Riemann Sphere

For many purposes it is useful to extend the set $\mathbb{C}$ of complex numbers by introducing a symbol ∞ to represent infinity.

To this end, consider the complex plane to which is appended a unit sphere centred at the origin.

For every point z in the complex plane, consider the line joining this point to the North pole of the sphere.

This line intersects the sphere in a unique point. If we designate this point by (ξ_1, ξ_2, ξ_3), where $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$, then using similar triangles we see that if $z = x + iy$,

$$x = \frac{\xi_1}{1-\xi_3} \quad \text{and} \quad y = \frac{\xi_2}{1-\xi_3} \ ,$$

$$z = \frac{\xi_1 + i\xi_2}{1-\xi_3}$$

From this we can calculate

$$\begin{aligned}
|z|^2 = x^2 + y^2 &= \frac{\xi_1^2 + \xi^2}{(1-\xi_3)^2} \\
&= \frac{1-\xi_3^2}{(1-\xi_3)^2} = \frac{1+\xi_3}{1-\xi_3} \\
|z|^2 - \xi_3|z|^2 &= 1 + \xi_3 \\
\xi_3 &= \frac{|z|^2 - 1}{|z|^2 + 1} \\
1 - \xi_3 &= \frac{2}{|z|^2 + 1} \\
\xi_1 &= \frac{2x}{|z|^2 + 1} \\
\xi_2 &= \frac{2y}{|z|^2 + 1}
\end{aligned}$$

This shows that the point on the sphere is uniquely determined.

Conversely, every point of the sphere except the North pole $(0, 0, 1)$ corresponds to a unique point in $\mathbb{C}$.

This $1-1$ mapping is called a **stereographic projection**.

As $|z| \to \infty$, we see that $\xi_1 \to 0$, $\xi_2 \to 0$ and $\xi_3 \to 1$. Hence we associate the North pole of the sphere with the complex **point at infinity**.

This representation of the complex numbers plus ∞ is referred to as the **Riemann sphere**.

If $z = x + iy$ maps onto (ξ_1, ξ_2, ξ_3), then the diametrically opposite point $(-\xi_1, -\xi_2, -\xi_3)$ corresponds to

$$\frac{-\xi_1 - i\xi_2}{1+\xi_3} = -\frac{\xi_1 + i\xi_2}{1-\xi_3}\,\frac{1-\xi_3}{1+\xi_3} = -\frac{z}{|z|^2} = -\frac{1}{z^*}$$

Metric.

We measure the distance between points in the complex plane $\mathbb{C}$ by using the modulus function.

If $w = u + iv$ and $z = x + iy$ are two points in $\mathbb{C}$, then the distance $d(w, z)$ between them is defined to be

$$\begin{aligned} d(w, z) &= |w - z| \\ &= |(u - x) + i(v - y)| \\ &= \sqrt{(u - x)^2 + (v - y)^2} \end{aligned}$$

This distance function coincides with the usual Euclidean distance in $\mathbb{R}^2$.

It satisfies the four requirements of a metric:

1. $d(w, z) \geq 0$ for all w, z.
2. $d(w, z) = 0$ if and only if $w = z$.
3. $d(z, w) = d(w, z)$.
4. $d(\zeta, z) \leq d(\zeta, w) + d(w, z)$.

The first three of these properties are obvious from the definition, while the fourth property is a restatement of the *Triangle inequality*.

Exercise.

Show that if the Riemann sphere is used instead of the complex plane, the distance between z_1 and z_2 is

$$\frac{2|z_1 - z_2|}{\sqrt{(1 + |z_1|^2)(1 + |z_2|^2)}} \ .$$

What is the distance from z to ∞ in this metric?

Open and closed sets.

For a given point $z_0 \in \mathbb{C}$, the set of points satisfying

$$|z - z_0| < \epsilon \ , \ \epsilon > 0$$

is called an *open disc*, of radius ϵ, centered at z_0 or the *epsilon neighbourhood of* z_0.

An arbitrary set $S \subset \mathbb{C}$ is said to be **open** (in $\mathbb{C}$) if for every element $\zeta \in S$ there is an open disc centered at ζ all of whose points are contained in S.

In particular, an open disc is an open set.

Let $S = \{|z - z_0| < \epsilon\}$, and consider any $\zeta \in S$. Then $|\zeta - z_0| = r < \epsilon$.

Let $\epsilon_1 = \epsilon - r > 0$.

If $|z - \zeta| < \epsilon_1$, then

$$|z - z_0| = |z - \zeta + \zeta - z_0| \leq |z - \zeta| + |\zeta - z_0| < \epsilon_1 + r = \epsilon \ .$$

Therefore the open disc of radius ϵ_1 centered at ζ is contained in S, and S is open.

Boundary points. A point ζ is a *boundary point* of the set S if every neighbourhood of ζ contains at least one point which is in S and at least one point which is not in S. The boundary point may or may not be in the set S.

Note that an open set does not contain any of its boundary points.

For example, the boundary points of the open disc of radius ϵ about z_0 are the points on the circle $|z - z_0| = \epsilon$.

Definition. Given a set S of numbers, the number l is called a **limit point** or **accumulation point** of the set, if, given any $\epsilon > 0$ there is a number $z \neq l$ in S such that $|z - l| < \epsilon$.

The limit point may or may not be a member of S.

If we start with $\epsilon_1 > 0$ and determine such a number $z_1 \in S$, then we can define $\epsilon_2 = |z_1 - l|$.

Since $z_1 \neq l$, $\epsilon_2 > 0$, and we can determine a number $z_2 (\neq l) \in S$ such that

$$|z_2 - l| < |z_1 - l| < \epsilon_1 \ .$$

Note that $z_2 \neq z_1$.

Continuing in this way, we see that if l is a limit point, then for any $\epsilon > 0$ there are infinitely many points z in S such that $|z - l| < \epsilon$.

Conversely, if every neighbourhood of l contains infinitely many points of S, it contains at least one point of S which differs from l. Therefore l is a limit point.

A set S which contains all its limit points is said to be **closed**.

In particular, the set

$$\{|z - z_0| \leq \epsilon\}$$

is a closed disc of radius epsilon centered at z_0.

Open discs occur naturally when we consider limits in $\mathbb{C}$.

Given a sequence $\{z_n\}$ of numbers in $\mathbb{C}$, we say that this sequence converges to l if for every (real) $\epsilon > 0$, there is a natural number N such that,

$$|z_n - l| < \epsilon \quad \text{for every natural number} \quad n > N \ .$$

That is, every term in the sequence from z_{N+1} onwards lies in the open disc of radius ϵ centred at l.

This definition of convergence has the same structure as the definition of convergence used for real sequences, and consequently the basic limit theorems of real analysis carry over for complex sequences also.

For example:

1. If $\{z_n\}$ converges to l and $\{\zeta_n\}$ converges to λ, then $\{z_n + \zeta_n\}$ converges to $l + \lambda$.

Proof: Given $\epsilon > 0$, there is a natural number N_1 such that

$$|z_n - l| < \frac{\epsilon}{2} \quad \text{for all} \quad n > N_1$$

and there is a natural number N_2 such that

$$|\zeta_n - \lambda| < \frac{\epsilon}{2} \quad \text{for all} \quad n > N_2 \ .$$

Then, for $n > N = \max(N_1, N_2)$,

$$\begin{aligned} |(z_n + \zeta_n) - (l + \lambda)| &= |(z_n - l) + (\zeta_n - \lambda)| \\ &\leq |z_n - l| + |\zeta_n - \lambda| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \end{aligned}$$

2. If $\{z_n\}$ converges to l, then for any constant $a \in \mathbb{C}$, $\{a z_n\}$ converges to al.

Proof: If $a = 0$, $az_n = 0$ for all n, and $al = 0$ also. Then for any $\epsilon > 0$,

$$|az_n - al| = |0 - 0| = 0 < \epsilon$$

for every natural number n.

Otherwise, given $\epsilon > 0$, there is a natural number N such that

$$|z_n - l| < \frac{\epsilon}{|a|} \quad \text{for all} \quad n > N \ .$$

Then

$$|az_n - al| = |a|\,|z_n - l| < \epsilon \quad \text{for all} \quad n > N \ .$$

Finally, the sequence $\{z_n = x_n + iy_n\}$ converges to $l = l_1 + il_2$ if and only if the real sequences $\{x_n\}$ and $\{y_n\}$ converge to l_1 and l_2 respectively.

The **if** part follows from the first two results.

If $\{y_n\}$ converges to l_2, then $\{iy_n\}$ converges to il_2, and $\{x_n + iy_n\}$ converges to $l_1 + il_2$.

For the **only if** part we note that

$$|z_n - l|^2 = (x_n - l_1)^2 + (y_n - l_2)^2 = |x_n - l_1|^2 + |y_n - l_2|^2$$

so that

$$\begin{aligned} & |x_n - l_1| \le |z_n - l| \\ \text{and} \quad & |y_n - l_2| \le |z_n - l| \ . \end{aligned}$$

If $\{z_n\}$ converges to l, then given any $\epsilon > 0$ there is a natural number N such that

$$|z_n - l| < \epsilon \quad \text{for all} \quad n > N \ .$$

But then

$$|x_n - l_1| < \epsilon \quad \text{for all} \quad n > N$$

and

$$|y_n - l_2| < \epsilon \quad \text{for all} \quad n > N \ ,$$

and the sequences converge as required.

As an application of this result, consider the Geometric progression

$$S_n = 1 + z + z^2 + \cdots + z^{n-1} = \frac{1 - z^n}{1 - z} \ .$$

For $|z| < 1$, the sequence $\{S_n\}$ converges to $1/(1 - z)$.

Setting $z = re^{i\theta}$, where $0 \le r < 1$, we have

$$\begin{aligned} z^k &= r^k e^{ik\theta} = r^k \cos k\theta + ir^k \sin k\theta \\ Re(S_n) &= 1 + r\cos\theta + r^2 \cos 2\theta + \cdots + r^{n-1}\cos(n-1)\theta \\ Im(S_n) &= r\sin\theta + r^2 \sin 2\theta + \cdots + r^{n-1}\sin(n-1)\theta \\ \frac{1}{1-z} &= \frac{1 - z^*}{1 - (z + z^*) + zz^*} = \frac{1 - r\cos\theta + ir\sin\theta}{1 - 2r\cos\theta + r^2} \end{aligned}$$

so that, for $0 \le r < 1$,

$$\sum_{k=0}^{\infty} r^k \cos k\theta = \frac{1 - r\cos\theta}{1 - 2r\cos\theta + r^2}$$
$$\sum_{k=1}^{\infty} r^k \sin k\theta = \frac{r\sin\theta}{1 - 2r\cos\theta + r^2}$$

Cauchy sequences.

The disadvantage of the definition of convergence, which is shared by real sequences, is that the test is **extrinsic**. In order to apply it to any given sequence

$$\text{e.g.} \qquad z_n = \left(1 + \frac{i}{n}\right)^n$$

it is necessary to know the value of the limit l.

However, if $\{z_n\}$ converges to l, then given $\epsilon > 0$, there is a natural number N such that

$$|z_n - l| < \frac{1}{2}\epsilon \quad \text{for all} \quad n > N \ .$$

If m is any other number greater than N, then

$$|z_m - l| < \frac{1}{2}\epsilon \quad \text{also.}$$

Therefore, for all values of m and n greater than N,

$$\begin{aligned} |z_m - z_n| &= |(z_m - l) - (z_n - l)| \\ &\le |z_m - l| + |z_n - l| < \epsilon \end{aligned}$$

This condition

$$\text{Given any } \epsilon > 0 \ , \ \exists \ N \ ; \ |z_m - z_n| < \epsilon \ \ \forall \ m, n > N$$

is called **Cauchy's criterion**, and a sequence for which it is true is called a **Cauchy sequence**.

We have shown that a convergent sequence is a Cauchy sequence.

It is also true that a Cauchy sequence in $\mathbb{C}$ is convergent.

The proof of this result is given in the appendix

Cauchy's criterion provides an **intrinsic** test for convergence; that is, a test which depends only on knowing the terms of the sequence.

One application of this result is in the **absolute convergence** of infinite series.

Given an infinite sum

$$\sum_{k=0}^{\infty} A_k$$

where the terms A_k are complex numbers, we consider instead the sum

$$\sum_{k=0}^{\infty} |A_k|$$

of non-negative real numbers.

If this sum of absolute values converges, then the sequence of partial sums $\{S_n\}$, where

$$S_n = \sum_{k=0}^{n} |A_k|$$

is a Cauchy sequence.

Therefore, given any $\epsilon > 0$, there is an integer N such that, if $n > m > N$, $|S_n - S_m| < \epsilon$.

Since the terms in the sum are non-negative,

$$|S_n - S_m| = \sum_{k=m+1}^{n} |A_k| \ .$$

If we now consider the corresponding partial sums s_n and s_m of the original series,

$$\begin{aligned} |s_n - s_m| &= \left| \sum_{k=m+1}^{n} A_k \right| \\ &\leq \sum_{k=m+1}^{n} |A_k| \quad \text{(the triangle inequality)} \\ &< \epsilon \quad \text{for all} \quad n > m > N \ . \end{aligned}$$

Therefore the sequence of partial sums $\{s_n\}$ is a Cauchy sequence, and so converges in $\mathbb{C}$.

When this test is successful, we say that the infinite sum **converges absolutely** or is **absolutely convergent**.

However, as you will be aware, not all convergent sums are absolutely convergent. The most famous counter example is the alternating sum

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

which converges to $\log 2$, while the corresponding sum of absolute values is the harmonic series

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$$

which diverges.

We will return to this topic later, when we consider Taylor's Theorem and power series expansions.