

SPECIAL LINEAR FRACTIONAL TRANSFORMATIONS

The linear fractional transformation

$$w = \frac{az + b}{cz + d}$$

involves three independent constants.

This means that the transform is completely determined if we specify three pairs

$$z_i \rightarrow w_i$$

of points and their maps.

The appropriate transformation can be derived by considering the multi-step process

$$\begin{aligned} z_1 &\rightarrow 0 \rightarrow w_1 \\ z_2 &\rightarrow \infty \rightarrow w_2 \end{aligned}$$

Denoting the intermediate variable by ζ , we have

$$\begin{aligned} \zeta &= c_1 \frac{z - z_1}{z - z_2} \\ \zeta &= c_2 \frac{w - w_1}{w - w_2} \\ \frac{w - w_1}{w - w_2} &= c \frac{z - z_1}{z - z_2} \end{aligned}$$

We can now use the third piece of data; $z_3 \rightarrow w_3$; to evaluate the constant.

$$\frac{w_3 - w_1}{w_3 - w_2} = c \frac{z_3 - z_1}{z_3 - z_2}$$

which gives the symmetric form

$$\frac{w - w_1}{w - w_2} \frac{w_3 - w_2}{w_3 - w_1} = \frac{z - z_1}{z - z_2} \frac{z_3 - z_2}{z_3 - z_1}$$

Note that if $z_4 \rightarrow w_4$ also then

$$\frac{w_4 - w_1}{w_4 - w_2} \frac{w_3 - w_2}{w_3 - w_1} = \frac{z_4 - z_1}{z_4 - z_2} \frac{z_3 - z_2}{z_3 - z_1}$$

This invariant quantity is known as the *cross ratio*.

e.g.

Given the pairs

$$\begin{aligned} 0 &\rightarrow 1 \\ 1 &\rightarrow i \\ i &\rightarrow 0 \\ &1 \end{aligned}$$

the bilinear transformation which embodies these specific mappings is

$$\begin{aligned} \frac{w-1}{w-i} \frac{0-i}{0-1} &= \frac{z-0}{z-1} \frac{i-1}{i-0} \\ i \frac{w-1}{w-i} &= (1+i) \frac{z}{z-1} \\ i(w-1)(z-1) &= (1+i)z(w-i) \\ iwz - iz - iw + i &= (1+i)wz + (1-i)z \\ wz + iw &= -z + i \\ w &= -\frac{z-i}{z+i} \end{aligned}$$

MAPPING THE UPPER HALF PLANE ONTO THE INTERIOR OF THE UNIT CIRCLE

To determine the general form of such a mapping we proceed in stages.

1. Some point z_0 in the upper half plane maps onto $w = 0$.

Therefore the function is of the form

$$w = \frac{z - z_0}{cz + d}$$

for some c, d .

2. The point at infinity ($z = \infty$) maps onto some point $w = e^{i\theta}$ on the unit circle.

$$\frac{z - z_0}{cz + d} = \frac{1 - z_0/z}{c + d/z} \rightarrow \frac{1}{c}$$

therefore

$$\frac{1}{c} = e^{i\theta}$$

and

$$\frac{z - z_0}{cz + d} = e^{i\theta} \frac{z - z_0}{z - \zeta_0}$$

where $\zeta_0 = -de^{i\theta}$.

3. Every point $z = x$ maps onto $|w| = 1$.

Setting $z_0 = x_0 + iy_0$ and $\zeta_0 = \xi_0 + i\eta_0$, we have

$$\begin{aligned} 1 &= |e^{i\theta}| \frac{|x - x_0 - iy_0|}{|x - \xi_0 - i\eta_0|} \\ |x - x_0 - iy_0| &= |x - \xi_0 - i\eta_0| \\ (x - x_0)^2 + y_0^2 &= (x - \xi_0)^2 + \eta_0^2 \\ x_0 &= \xi_0 \\ y_0^2 &= \eta_0^2 \end{aligned}$$

If we choose $\eta_0 = y_0$, the function reduces to the constant $e^{i\theta}$.

Therefore $\eta = -y_0$, $\zeta_0 = z_0^*$, and the required mapping is

$$w = e^{i\theta} \frac{z - z_0}{z - z_0^*}$$

with $y_0 > 0$.

MAPPING THE UNIT CIRCLE ONTO ITSELF

We can achieve this mapping by first taking a specific mapping of the unit circle onto the upper half plane, and then taking a general mapping of the complex plane onto the unit circle.

The first stage is the inverse of the mapping

$$z = \frac{\zeta - i}{\zeta + i}$$

namely

$$\zeta = \frac{iz + i}{-z + 1}$$

Now if

$$w = e^{i\theta} \frac{\zeta - a}{\zeta - a^*}$$

we can substitute for ζ to obtain

$$\begin{aligned} e^{-i\theta} w &= \frac{\frac{iz+i}{-z+1} - a}{\frac{iz+i}{-z+1} - a^*} \\ &= \frac{iz + i + az - a}{iz + i + a^*z - a^*} \\ &= \frac{(a+i)z - (a-i)}{(a^*+i)z - (a^*-i)} \\ &= \frac{a+i}{a^*-i} \frac{z - \frac{a-i}{a+i}}{\frac{a^*+i}{a^*-i}z - 1} \end{aligned}$$

Since a is in the upper half plane,

$$z_0 = \frac{a-i}{a+i}$$

is (an arbitrary point) inside the unit circle.

Also,

$$\frac{a^*+i}{a^*-i} = z_0^*$$

Finally

$$\begin{aligned} a^* - i &= (a+i)^* \\ |a^* - i| &= |a+i| \\ \frac{a+i}{a^*-i} &= e^{i\phi} \end{aligned}$$

Putting all this together we get

$$w = e^{i\alpha} \frac{z - z_0}{z_0^*z - 1}$$

where z_0 is an arbitrary point inside the unit circle, and $\alpha = \theta + \phi$ is an arbitrary angle.

e.g.

Find the Möbius transformation which maps the unit circle onto itself, which maps $z = \frac{1}{2}$ onto $w = 0$ and which maps $z = -1$ onto $w = -1$.

Since $\frac{1}{2} \rightarrow 0$, $z_0 = \frac{1}{2}$, and we have

$$w = e^{i\alpha} \frac{z - \frac{1}{2}}{\frac{1}{2}z - 1}$$

Now if $-1 \rightarrow -1$,

$$\begin{aligned} -1 &= e^{i\alpha} \frac{-1 - \frac{1}{2}}{-\frac{1}{2} - 1} = e^{i\alpha} \\ w &= -\frac{z - \frac{1}{2}}{\frac{1}{2}z - 1} = \frac{2z - 1}{2 - z} \end{aligned}$$

It is easy to see that $\frac{1}{2} \rightarrow 0$ and $-1 \rightarrow -1$.

As a final check, consider $z = e^{i\theta}$.

$$\begin{aligned} |2z - 1|^2 &= |2 \cos \theta - 1 + 2i \sin \theta|^2 \\ &= (2 \cos \theta - 1)^2 + 4 \sin^2 \theta \\ &= 4 \cos^2 \theta - 4 \cos \theta + 1 + 4 \sin^2 \theta \\ &= 5 - 4 \cos \theta \\ |2 - z|^2 &= |2 - \cos \theta - i \sin \theta|^2 \\ &= (2 - \cos \theta)^2 + \sin^2 \theta \\ &= 4 - 4 \cos \theta + \cos^2 \theta + \sin^2 \theta \\ &= 5 - 4 \cos \theta \end{aligned}$$

i.e. When $z = e^{i\theta}$, $|w| = 1$ as required.

MAPPING THE REGION BETWEEN TWO NON-INTERSECTING CIRCLES
ONTO THE ANNULAR REGION BETWEEN TWO CONCENTRIC CIRCLES

Let C_1 and C_2 be the circles in the z plane.

We begin by reducing this general problem to a *standard form*.

If the circles lie one inside the other, let C_2 be the larger circle. Otherwise, we can label the circles arbitrarily.

By means of a translation, we can move the centre of C_2 to the origin.

By means of a dilation we can make the radius of C_2 equal to 1.

If C_1 lies inside C_2 , this step is not required. If C_1 lies outside C_2 , then the inversion

$$w = \frac{1}{z}$$

leaves C_2 in situ, but shifts C_1 inside C_2 .

Finally, we can rotate the plane so that the centre of C_1 lies on the positive real axis.

The problem now reduces to that of finding a Möbius transformation which will transform the region between the circle

$$C_1 = |z - a| = r$$

where $0 < a < a+r < 1$, and the unit circle, onto the region between two concentric circles.

Specifically, we will map C_2 onto a unit circle in the w plane, and C_1 onto a circle $|w| = \rho < 1$.

This means that we will have a mapping of the form

$$w = e^{i\theta} \frac{z - z_0}{z_0^* z - 1}$$

If we consider the common diameter of the circles C_1 and C_2 , which intersects each circle at right angles, we see that it must map into a common diameter of the new circles.

Therefore the point which maps onto the origin lies on the real axis in the z plane.

Also, the final figure is invariant under rotations, so that we can choose $e^{i\theta} = -1$.

Therefore

$$w = \frac{z - c}{1 - cz}$$

for some real c .

The inverse transformation is

$$z = \frac{w + c}{cw + 1}$$

C_1 maps onto

$$\begin{aligned} \left| \frac{w+c}{cw+1} - a \right| &= r \\ |w+c - a(cw+1)| &= r|cw+1| \\ |(1-ac)w + (c-a)| &= |rcw+r| \\ (1-ac)^2 ww^* + (1-ac)(c-a)(w+w^*) + (c-a)^2 \\ &= r^2 c^2 ww^* + r^2 c(w+w^*) + r^2 \end{aligned}$$

If this is a circle centred at the origin, then

$$\begin{aligned} (1-ac)(c-a) &= r^2 c \\ ac^2 + (r^2 - a^2 - 1)c + a &= 0 \end{aligned}$$

Alternatively, we can derive this equation by noting that the points of intersection with the common diameter maps onto similar points on the real axis in the w plane. This means $a+r \rightarrow \rho$ and $a-r \rightarrow -\rho$ so that

$$\frac{a+r-c}{1-ac-rc} + \frac{a-r+c}{1-ac+rc} = 0$$

This equation has two real roots, one of which has modulus less than 1. This is the required value of c .

e.g.

Find a Möbius transformation which maps the region between the circles $|z-2i|=1$ and $|z+2|=\frac{1}{2}$ onto the region between two concentric circles.

Since these circles are disjoint, we can choose C_2 to be $|z+2|=\frac{1}{2}$.

The translation

$$\zeta_1 = z + 2$$

gives

$$\begin{aligned} C_1 : \quad |\zeta_1 - 2 - 2i| &= 1 \\ C_2 : \quad |\zeta_1| &= \frac{1}{2} \end{aligned}$$

The dilation

$$\zeta_2 = 2\zeta_1 (= 2z + 4)$$

gives

$$\begin{aligned} C_1 : \quad |\zeta_2 - 4 - 4i| &= 2 \\ C_2 : \quad |\zeta_2| &= 1 \end{aligned}$$

The inversion

$$\zeta_3 = \frac{1}{\zeta_2} \left(= \frac{1}{2z+4} \right)$$

gives

$$\begin{aligned} \zeta_2 \zeta_2^* - (4+4i)\zeta_2^* - (4-4i)\zeta_2 + 32 &= 4 \\ 28\zeta_3 \zeta_3^* - (4-4i)\zeta_3^* - (4+4i)\zeta_3 + 1 &= 0 \\ \zeta_3 \zeta_3^* - \frac{1-i}{7}\zeta_3^* - \frac{1+i}{7}\zeta_3 + \frac{2}{49} \\ &= -\frac{1}{28} + \frac{2}{49} = \frac{1}{196} \end{aligned}$$

$$\begin{aligned} C_1 : \quad \left| \zeta_3 - \frac{1-i}{7} \right| &= \frac{1}{14} \\ C_2 : \quad |\zeta_3| &= 1 \end{aligned}$$

Since

$$\frac{1-i}{7} = \frac{\sqrt{2}}{7} e^{-i\pi/4}$$

we choose the rotation

$$\zeta_4 = e^{i\pi/4} \zeta_3 \left(= \frac{e^{i\pi/4}}{2z+4} \right)$$

which gives

$$\begin{aligned} C_1 : \quad \left| \zeta_4 - \frac{\sqrt{2}}{7} \right| &= \frac{1}{14} \\ C_2 : \quad |\zeta_4| &= 1 \end{aligned}$$

Finally we have the mapping

$$w = \frac{\zeta_4 - c}{1 - c\zeta_4}$$

where

$$\begin{aligned} \frac{\sqrt{2}}{7} c^2 - \left(1 + \frac{2}{49} - \frac{1}{196} \right) c + \frac{\sqrt{2}}{7} &= 0 \\ c^2 - \frac{25}{4\sqrt{2}} c + 1 &= 0 \\ c = \frac{25 - \sqrt{497}}{8\sqrt{2}} &\sim 0.239 \end{aligned}$$

This mapping gives

$$\begin{aligned} C_1 : \quad |w| &= .0366 \\ C_2 : \quad |w| &= 1 \end{aligned}$$