Complex Analysis

INTRODUCTION

The solutions of the quadratic equation

$$ax^2 + bx + c = 0$$

can be written as

$$x = (-b \pm \sqrt{(b^2 - 4ac)})/2a\ .$$

Where the problem which gives rise to the equation requires real solutions, we say that the problem has no solutions if $b^2 - 4ac < 0$.

(Equally, if the original problem requires rational solutions, we would say that the equation has no solutions unless $\sqrt{(b^2 - 4ac)}$ is rational.)

However, in some cases we have no such information about the type of solution that is valid, and then we must accept that expressions involving square roots of negative numbers may be possible solutions.

Historically, this situation arose when a formula, similar to that given above for the quadratic equation, was sought for the solution of the general cubic equation.

To solve

$$x^3 = 3px + 2q$$

(The 3 and 2 are inserted to keep the algebraic expressions simple.)

we set

$$x = w + \frac{p}{w}\ .$$

The equation now reduces to

$$w^6 - 2qw^3 + p^3 = 0\ ,$$

which is a quadratic equation in w^3, giving

$$\begin{aligned} w^3 &= q + \sqrt{(q^2 - p^3)} \\ w &= (q + \sqrt{(q^2 - p^3)})^{1/3} \\ x &= (q + \sqrt{(q^2 - p^3)})^{1/3} + \frac{p}{(q + \sqrt{(q^2 - p^3)})^{1/3}} \\ x &= (q + \sqrt{(q^2 - p^3)})^{1/3} + (q - \sqrt{(q^2 - p^3)})^{1/3} \end{aligned}$$

(Note that the solution $w^3 = q - \sqrt{(q^2 - p^3)}$ gives the same value for x.)

Now, even in the very simple case $p = 1$, $q = 0$, for which the three solutions of the cubic equation

$$x^3 = 3x$$

are $x = 0, \sqrt{3}, -\sqrt{3}$, this general method gives

$$w^3 = \sqrt{(-1)}\ ,$$

indicating that we may need to use square roots of negative numbers to obtain the real solutions of a real problem.

This equation and its solution raise three questions.

1. What are the rules for manipulating expressions involving square roots of negative numbers?

2. The original equation has three real solutions, but, for real numbers at least, the cube root has a unique real value. How then can we find three solutions?

3. We have already had to consider square roots of negative numbers. Will taking cube roots involve us in yet more complication?

(The answer to this question is, fortunately, 'No'.)

Exercises.

1. Find a solution of the equation

$$x^3 + 6x = 2 \ .$$

2. Show that the substitution $y = x - a$ reduces the equation

$$y^3 + 3ay^2 + 3by + c = 0$$

to the form

$$x^3 = 3(a^2 - b)x + (3ab - 2a^3 - c) \ .$$

Hence find a solution of the equation

$$y^3 = 3y^2 + 3y + 1 \ .$$

Complex Numbers

We introduce the symbol[1] i, and assign to it the property that

$$i^2 = -1 \ .$$

A **complex number** is then defined as an expression of the form $a+bi$ or $a+ib$, where a and b are real numbers.

(These numbers are called complex, not because they are difficult to handle, but because they have more than one component. c.f. "shopping complex")

If $b = 0$, the complex number $a + bi$ reduces to the real number a, so that we can consider real numbers as a special case of complex numbers.

For a general complex number $a+bi$, we call 'a' the real part (denoted $Re(a+bi)$) and 'b' the imaginary part (denoted $Im(a + bi)$).

Note that, in spite of its name, the imaginary part of $a + bi$ is real.

We say that two complex numbers are equal if and only if their real parts are equal and their imaginary parts are equal.

The rules for manipulating complex numbers follow from the corresponding rules for real numbers.

Thus

$$(a + bi) + (c + di) = (a + c) + (b + d)i$$

and

$$\begin{aligned}(a + bi)(c + di) &= ac + adi + bci + bdi^2 \\ &= (ac - bd) + (ad + bc)i\end{aligned}$$

since $i^2 = -1$.

[1] Engineers prefer to use j.

Under these rules the set of complex numbers is closed under addition and multiplication.

It can be shown that complex numbers satisfy the associative and commutative laws for addition and multiplication, and the distributive law.

The zero complex number is $0+0i=0$ and the multiplicative unit is $1+0i=1$.

The multiplicative inverse of $a+bi \neq 0$ can be found by the same method as that used in school for simplifying surds;

$$\begin{aligned}\frac{1}{a+bi} &= \frac{1}{a+bi}\,\frac{a-bi}{a-bi} \\ &= \frac{a-bi}{(a^2+b^2)+(ab-ab)i} \\ &= \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i\end{aligned}$$

These rules mean that the complex numbers form a field.

Notice also that

$$(0+bi)(0+bi) = -b^2 \ ,$$

so that the square root of any negative real number can be expressed in terms of i.

Exercises.

1. Verify the distributive law for complex numbers.

2. Express in the form $a+bi$

(a) $$(1-i)(2+3i)$$

(b) $$\frac{1}{1+2i}$$

(c) $$\frac{2-3i}{1+2i}$$

(d) $$\frac{1}{(1+i)^2} + \frac{1}{(1-i)^2}$$

(e) $$\frac{1}{\cos\theta + i\,\sin\theta}$$

3. By factorizing x^3-1 as $(x-1)(x^2+x+1)$, find three distinct solutions of the equation $x^3=1$.

Use this answer to find three distinct solutions of the equation $x^3=\alpha^3$, $\alpha \neq 0$.

Hence find three solutions for

$$x^3+6x=2$$

and

$$y^3 = 3y^2+3y+1 \ .$$

(See the previous exercises)

4. Evaluate $(-i)^3$, and hence use the general formula for the solution of a cubic equation and question 3 to derive the solutions

$$x = 0\,, \sqrt{3}\,, -\sqrt{3}$$

of the equation $x^3=3x$.

Geometrical representation of complex numbers
The Argand diagram

Just as we may represent real numbers by points on a line or by directed line segments joining the origin to these points, so we may represent complex numbers by points in a plane. This representation is referred to as an **Argand diagram** or **the complex plane**.

Taking rectangular axes with origin O, we may represent the complex number $z = x+iy$ by the point P whose coordinates are (x, y), or equivalently by the vector $\vec{OP}$.

The x-axis is called the **real axis**, and the y-axis is called the **imaginary axis**.

Addition then corresponds to the usual parallelogram law for the addition of vectors.

Multiplication is more easily described by using polar notation.

The length r of the vector $\vec{OP}$ is called the **modulus** of the complex number z, written

$$|z| = r\ .$$

($|z|$ is pronounced *mod* z.)

By Pythagoras' Theorem,

$$|z| = \sqrt{x^2 + y^2}\ .$$

The modulus is always greater than or equal to zero, and is zero if and only if $z = 0$.

If $z \neq 0$, we can consider the angle θ between the vector $\vec{OP}$ and the positive x axis. This angle is called the **argument** or **amplitude** of z, and denoted

$$\arg z$$

or

$$\operatorname{amp} z\ .$$

(These terms are interchangeable, but *argument* is more frequently used.)

The argument is not uniquely defined. If θ is an argument for z, so is $\theta + 2k\pi$ for any integer k.

(Note that we use radian measure for the angles.)

When we are evaluating $z = r\cos\theta + ir\sin\theta$, all values of the argument give the same value for z. However, as we shall see, in some applications where the solutions are not unique we need to consider different values of the argument.

One of these values satisfies

$$-\pi < \arg z \leq \pi\ ,$$

and we call this value the **principal value** of the argument[2].

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[2]Some people use $0 \leq \theta < 2\pi$ for the principal value.

Given x and y, the values of θ are determined (to within $2k\pi$) by the equations

$$\sin\theta = \frac{y}{r} \quad , \quad \cos\theta = \frac{x}{r} \; .$$

One of these equations will determine two values of θ with $-\pi < \theta \leq \pi$, and the second equation identifies which of these values is correct.

Alternatively, you can use

$$\tan\theta = \frac{y}{x}$$

to identify two possible values. Again you need one of the other equations to identify the correct choice.

Example 1.

Find the modulus and argument of $1 + i$.

Solution: Note that $x = 1$ and $y = 1$.

$$r = \sqrt{(1^2 + 1^2)} = \sqrt{2}$$

$$\tan\theta = \frac{1}{1} = 1 \; ; \; \theta = \frac{\pi}{4} \text{ or } -\frac{3\pi}{4}$$

However,

$$\sin\theta = \frac{y}{r} = \frac{1}{\sqrt{2}} > 0$$

so that $0 < \theta < \pi$.

Hence

$$\theta = \frac{\pi}{4}$$

Example 2.

Find the modulus and argument of $-\sqrt{3} + i$.

Solution: Note that $x = -\sqrt{3}$ and $y = 1$.

$$r = \sqrt{((-\sqrt{3})^2 + 1^2)} = \sqrt{(3+1)} = 2$$

$$\sin\theta = \frac{y}{r} = \frac{1}{2} \; ; \; \theta = \frac{\pi}{6} \text{ or } \frac{5\pi}{6}$$

Since

$$\cos\theta = \frac{x}{r} = -\frac{\sqrt{3}}{2} < 0$$

$$\theta = \frac{5\pi}{6} \; .$$

Exercises.

Determine the modulus and argument of the following complex numbers.

(1) $1 + \sqrt{3}i$

(2) $2i$

(3) $\left(\frac{1+i}{1-i}\right)^2$

We can now write a non-zero complex number in the form

$$z = x + iy = r(\cos\theta + i\sin\theta) ,$$

where $r > 0$.

When $z = 0$, we still have a modulus; $|z| = 0$; but the argument is not defined.

If $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 \left((\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\right) \\ &= r_1 r_2 \left(\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)\right) \end{aligned}$$

Thus

$$|z_1 z_2| = r_1 r_2 = |z_1|\,|z_2| .$$

In words, this states that the modulus of the product of two non-zero complex numbers is equal to the product of their moduli. This result is also trivially true if either (or both) of the numbers is zero.

We also see that $\theta_1 + \theta_2$ is an argument of $z_1 z_2$. That is, the argument of the product of two non-zero complex numbers is the sum of their arguments. (However, even if θ_1 and θ_2 are principal values, $\theta_1 + \theta_2$ may not be.)

Example.

$$\begin{aligned} 1 + i &= \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right) \\ -\sqrt{3} + i &= 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) \\ (1+i)(-\sqrt{3}+i) &= 2\sqrt{2}\left(\cos\left(\frac{\pi}{4} + \frac{5\pi}{6}\right) + i\sin\left(\frac{\pi}{4} + \frac{5\pi}{6}\right)\right) \\ (-1-\sqrt{3}) + (1-\sqrt{3})i &= 2\sqrt{2}\left(\cos\left(\frac{13\pi}{12}\right) + i\sin\left(\frac{13\pi}{12}\right)\right) \end{aligned}$$

In this case the principal argument is $-\frac{11\pi}{12}$.

If we write $z_1 z_2 = z_0$, we have

$$z_2 = \frac{z_0}{z_1} ,$$

and the above results become

$$\begin{aligned} \left|\frac{z_0}{z_1}\right| &= \frac{|z_0|}{|z_1|} \\ \arg\left(\frac{z_0}{z_1}\right) &= \arg(z_0) - \arg(z_1) . \end{aligned}$$

The complex conjugate

The number $x - yi$, where x and y are real, is called the **complex conjugate** of $z = x + yi$.

It is denoted either by
$$z^* = x - yi$$
or
$$\bar{z} = x - yi \ .$$

Note that

$$zz^* = (x + yi)(x - yi) = x^2 + y^2 = |z|^2$$
$$z + z^* = 2x = 2Re(z)$$
$$z - z^* = 2iy = 2iIm(z)$$
$$|z^*| = \sqrt{(x^2 + y^2)} = |z|$$
$$z_1^* z_2^* = (x_1 - y_1 i)(x_2 - y_2 i)$$
$$= x_1 x_2 - y_1 y_2 - (y_1 x_2 + y_2 x_1)i = (z_1 z_2)^*$$
$$\left(\frac{z_1}{z_2}\right)^* = \frac{z_1^*}{z_2^*}$$
$$z = z^* \text{ if and only if } z \text{ is real}$$

Geometrically, z^* is the reflection of z in the x-axis.

If z and w are complex numbers, then

$$\begin{aligned} |z + w|^2 &= (z + w)(z^* + w^*) \\ &= zz^* + (zw^* + z^*w) + ww^* \\ &= |z|^2 + 2Re(zw^*) + |w|^2 \end{aligned}$$

Similarly
$$|z - w|^2 = |z|^2 - 2Re(zw^*) + |w|^2$$
so that
$$|z + w|^2 + |z - w|^2 = 2(|z|^2 + |w|^2) \ .$$

For any complex number a

$$-|a| \leq Re(a) \leq |a| \ ,$$

so that
$$Re(zw^*) \leq |zw^*| = |z||w^*| = |z||w| \ .$$

Therefore,
$$|z + w|^2 \leq |z|^2 + 2|z||w| + |w|^2 = (|z| + |w|)^2 \ ,$$
and since $|z| + |w| \geq 0$,
$$|z + w| \leq |z| + |w| \ .$$

This result is known as the **Triangle inequality**.

It can be extended to general sums:

$$\left|\sum_{k=1}^{n} z_k\right| \le \sum_{k=1}^{n} |z_k| \ ;$$

and provided the sums converge

$$\left|\sum_{k=1}^{\infty} z_k\right| \le \sum_{k=1}^{\infty} |z_k| \ .$$

In particular,

$$|x+iy| \le |x| + |iy| = |x| + |y| \ .$$

de Moivre's Theorem

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For a positive integer n, we define the powers of z in the usual way

$$z^1 = z$$
$$z^n = z\, z^{n-1}$$

and for $z \neq 0$ we extend this definition to negative powers in the usual way

$$z^0 = 1$$
$$z^{-1} = \frac{1}{z}$$
$$z^{-n} = (z^{-1})^n$$

From these definitions the familiar index laws

$$z^m\, z^n = z^{m+n}$$
$$(z^m)^n = z^{mn}$$

follow.

de Moivre's Theorem.
If n is an integer, then

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta) \ .$$

Proof:
(i) Suppose firstly that $n \geq 1$.
If $n = 1$, the result is obvious.
For $n > 1$, we use mathematical induction.
Suppose that the result is true for $n = k \geq 1$. That is;

$$(\cos\theta + i\sin\theta)^k = \cos(k\theta) + i\sin(k\theta) \ .$$

Then

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)\\ &= (\cos(k\theta) + i\sin(k\theta)(\cos\theta + i\sin\theta)\\ &= (\cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta) + i(\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta)\\ &= \cos((k+1)\theta) + i\sin((k+1)\theta)\ .\end{aligned}$$

Hence by the principal of mathematical induction, the result is true for all positive integers n

(ii) If $n = 0$, then

$$(\cos\theta + i\sin\theta)^0 = 1$$
$$\cos(0) + i\sin(0) = 1 + i0 = 1$$

so that the result is true.

(iii) For $n \leq -1$, we first show that it is true for $n = -1$.

$$\begin{aligned}(\cos\theta + i\sin\theta)^{-1} &= \frac{1}{\cos\theta + i\sin\theta}\\ &= \frac{\cos\theta - i\sin\theta}{\cos^2\theta + \sin^2\theta}\\ &= \cos\theta - i\sin\theta\\ &= \cos(-\theta) + i\sin(-\theta)\end{aligned}$$

Now, for $n = -m$, we apply part (i) to this result;

$$\begin{aligned}z^n = z^{-m} = (z^{-1})^m &= (\cos(-\theta) + i\sin(-\theta))^m\\ &= \cos(m(-\theta)) + i\sin(m(-\theta))\\ &= \cos(-m\theta) + i\sin(-m\theta)\\ &= \cos(n\theta) + i\sin(n\theta)\end{aligned}$$

The expression $\cos\theta + i\sin\theta$ is usually written as

$$\cos\theta + i\sin\theta = e^{i\theta}\ .$$

Important: In this formula, θ **must** be measured in radians!

Note that

$$\begin{aligned}e^{i\frac{\pi}{2}} &= \cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right) = i\ ,\\ e^{i\pi} &= \cos(\pi) + i\sin(\pi) = -1\ ,\\ e^{i\frac{3\pi}{2}} &= \cos\left(\frac{3\pi}{2}\right) + i\sin\left(\frac{3\pi}{2}\right) = -i\ ,\\ e^{i2\pi} &= \cos(2\pi) + i\sin(2\pi) = 1\ .\end{aligned}$$

In particular,

$$iz = e^{i\frac{\pi}{2}}z$$

so that multiplication by i corresponds to a rotation through a right angle in the complex plane.

With this notation, de Moivre's Theorem takes the form

$$\left(e^{i\theta}\right)^n = e^{in\theta}$$

Trigonometric functions

Since

$$e^{i\theta} = \cos\theta + i\sin\theta$$

and

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

These expressions simplify the derivation of some trigonometric formulae.
For example:

$$\begin{aligned}\cos^3\theta &= \frac{(e^{i\theta} + e^{-i\theta})^3}{8} \\ &= \frac{e^{i3\theta} + 3e^{i\theta} + 3e^{-i\theta} + e^{-i3\theta}}{8} \\ &= \frac{1}{4}\frac{e^{i3\theta} + e^{-i3\theta}}{2} + \frac{3}{4}\frac{e^{i\theta} + e^{-i\theta}}{2} \\ &= \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta\end{aligned}$$

Exercises.

1. Prove that

(a)
$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$

(b)
$$\sin^4\theta = \frac{3}{8} - \frac{1}{2}\cos 2\theta + \frac{1}{8}\cos 4\theta$$

(c)
$$8\sin^3\theta\cos\theta = 2\sin 2\theta - \sin 4\theta\ .$$

2. Express

$$1 + e^{i\theta} + e^{i2\theta} + \cdots + e^{i(n-1)\theta}$$

as the sum of a geometric progression.

By considering the real and imaginary parts of this expansion and of the sum, show that

(a)
$$1 + \cos\theta + \cos 2\theta + \cdots + \cos(n-1)\theta$$
$$= \frac{1}{2}\left(1 + \frac{\cos(n-1)\theta - \cos n\theta}{1 - \cos\theta}\right) = \frac{1}{2}\left(1 + \frac{\sin(n - \frac{1}{2})\theta}{\sin\frac{1}{2}\theta}\right)$$

(b)
$$\sin\theta + \sin 2\theta + \cdots + \sin(n-1)\theta$$
$$= \frac{1}{2}\frac{\sin\theta + \sin(n-1)\theta - \sin n\theta}{1 - \cos\theta} = \sin\left(\frac{n}{2}\theta\right)\frac{\cos\left(\frac{n}{2} - 1\right)\theta - \cos\frac{n}{s}\theta}{1 - \cos\theta}$$
$$= \frac{\sin\left(\frac{n}{2}\theta\right)\sin\left(\frac{n-1}{2}\theta\right)}{\sin\frac{1}{2}\theta}$$

Fractional powers and roots of unity

We now consider the equation

$$z^n = a$$

where n is a positive integer, and a is any complex number.

We may suppose that $a \neq 0$, since that case is trivial, and also that $n \geq 2$.

We express a in polar form;

$$a = \rho(\cos\phi + i\sin\phi) = \rho e^{i\phi}$$

and look for solutions in the same form;

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}\ .$$

This gives the equation

$$r^n(\cos n\theta + i\sin n\theta) = \rho(\cos\phi + i\sin\phi)$$

form which we derive

$$r^n = \rho$$

which has the unique positive real solution

$$r = \rho^{1/n}$$

and the coupled equations

$$\begin{aligned}\cos n\theta &= \cos\phi\\ \sin n\theta &= \sin\phi\end{aligned}$$

from which we obtain

$$\begin{aligned}n\theta &= \phi + 2k\pi\\ \theta &= \frac{\phi}{n} + \frac{2k\pi}{n}\end{aligned}$$

where k is any integer.

While we now have infinitely many values for θ, there are only n distinct values for z, corresponding to $k = 0, 1, \ldots, n-1$.

If $k = qn + s$, where $0 \leq s < n$, then

$$\begin{aligned}\cos\theta =\ & \cos\left(\frac{\phi}{n} + \frac{2k\pi}{n}\right)\\ =\ & \cos\left(\frac{\phi}{n} + \frac{2s\pi}{n} + 2q\pi\right)\\ =\ & \cos\left(\frac{\phi}{n} + \frac{2s\pi}{n}\right)\\ \text{and}\quad \sin\theta =\ & \sin\left(\frac{\phi}{n} + \frac{2k\pi}{n}\right)\\ =\ & \sin\left(\frac{\phi}{n} + \frac{2s\pi}{n} + 2q\pi\right)\\ =\ & \sin\left(\frac{\phi}{n} + \frac{2s\pi}{n}\right)\end{aligned}$$

All the solutions have the same modulus r, while their successive arguments differ by $\frac{2\pi}{n}$.

When plotted on an Argand diagram, these points form the vertices of a regular polygon with n sides.

The special case $a = 1$ is of particular importance.

The solutions of $z^n = 1$ are called **roots of unity**.

Since $|1| = 1$ and $\arg(1) = 0$, these roots are

$$z = e^{i\frac{2k\pi}{n}} = \left(e^{i\frac{2\pi}{n}}\right)^k$$

for $k = 0, 1, \ldots, n-1$.

If we set

$$\omega = e^{i\frac{2\pi}{n}} \ ,$$

then the solutions of the equation are

$$1,\ \omega,\ \omega^2,\ \ldots,\ \omega^{n-1}$$

Also, if ζ is any solution of $z^n = a$, then

$$\zeta,\ \zeta\omega,\ \zeta\omega^2, \ldots \zeta\omega^{n-1}$$

represent the complete set of solutions of this equation.

For example; if $\zeta^2 = a$, then the square roots of a are

$$\zeta \quad \text{and} \quad -\zeta\ ;$$

if $\zeta^3 = a$, then the cube roots of a are

$$\zeta, \quad \frac{-1+i\sqrt{3}}{2}\zeta \quad \text{and} \quad \frac{-1-i\sqrt{3}}{2}\zeta\ ;$$

and if $\zeta^4 = a$, then the fourth roots of a are

$$\zeta, \quad i\zeta, \quad -\zeta \quad \text{and} \quad -i\zeta\ .$$

Exercises.

Solve the following equations, and plot the solutions on an Argand diagram.

(a) $$z^3 = 18 + 26i$$

(b) $$z^4 = -7 - 24i$$

(c) $$z^6 = -117 + 44i$$

Returning to the roots of unity, we have

$$\begin{aligned} \omega^n - 1 &= 0 \\ (\omega - 1)(\omega^{n-1} + \cdots + \omega + 1) &= 0 \\ \text{and} \quad \omega^{n-1} + \omega^{n-2} + \cdots + \omega + 1 &= 0 \end{aligned}$$

since $\omega \neq 1$.

Clearly

$$1 + \alpha + \alpha^2 + \cdots + \alpha^{n-1} = 0$$

if α is **any** root of $z^n = 1$ other than $z = 1$. However, $1,\ \alpha,\ \ldots, \alpha^{n-1}$ do **not** necessarily give all the roots unless n is a prime.

Exercises.

1.Show that if α is any root other than 1 of the equation $z^5 = 1$, then all the roots of this equation are given by

$$1,\ \alpha,\ \alpha^2,\ \alpha^3,\ \alpha^4\ .$$

2. Simplify

$$\sum_{j=0}^{2} \omega^j \left(\sum_{k=0}^{2} z^{2-k} \omega^{-jk} \right)$$

where $\omega = e^{2\pi i/3}$.

The Discrete Fourier Transform

These properties of the roots of unity are exploited in the **discrete Fourier Transform**.

If $f(t)$ is a piecewise continuous, piecewise smooth function with period $2T$, we can find the Fourier expansion

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi t}{T}\right) + b_n \sin\left(\frac{n\pi t}{T}\right)\right)$$

where

$$a_n = \frac{1}{T}\int_0^{2T} f(t)\cos\left(\frac{n\pi t}{T}\right)\, dt$$
$$b_n = \frac{1}{T}\int_0^{2T} f(t)\sin\left(\frac{n\pi t}{T}\right)\, dt$$

which converges to

$$\frac{1}{2}\left(f(t+0) + f(t-0)\right)$$

on $(0, 2T)$, and to its periodic extension outside this interval. If f is continuous, piecewise smooth and periodic with period $2T$, then the series converges to f on $\mathbb{R}$.

This expansion represents the function in terms of the fundamental and harmonic frequencies on the interval $[0, 2T]$; the coefficients a_n and b_n giving the amplitudes of the components.

Using the formulae for the sine and cosine functions in terms of the complex exponential, (and setting $\theta = \pi t/T$), we can rewrite the expansion as

$$\begin{aligned}
&\frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left(a_n \frac{1}{2}\left(e^{in\theta} + e^{-in\theta}\right) + b_n \frac{1}{2i}\left(e^{in\theta} - e^{-in\theta}\right)\right) \\
&= \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \frac{a_n - ib_n}{2} e^{in\theta} + \frac{a_n + ib_n}{2} e^{-in\theta} \\
&= \sum_{-\infty}^{\infty} c_n e^{in\theta} \qquad = \sum_{-\infty}^{\infty} c_n e^{in\pi t/T}
\end{aligned}$$

where

$$\begin{aligned} c_n &= \frac{1}{2}a_n - i\frac{1}{2}b_n \\ &= \frac{1}{2T}\int_0^{2T} f(t)\left(\cos(n\theta) - i\sin(n\theta)\right)\,dt \\ &= \frac{1}{2T}\int_0^{2T} f(t)e^{-in\pi t/T}\,dt \end{aligned}$$

Suppose now that our knowledge of the function f is obtained by uniformly spaced sampling the function N times over the interval $[0, 2T)$; that is, with $h = 2T/N$, we know

$$f(0), f(h), f(2h), \ldots f((N-1)h) \quad \text{and} \quad f(Nh) = f(0)\ .$$

We can estimate the coefficients c_n by using the trapezoidal rule for numerical integration

$$\begin{aligned} c_n &\sim \frac{1}{2T}\frac{h}{2}\left(f(0) + 2\sum_{k=1}^{N-1} f(kh)e^{-in\pi kh/T} + f(Nh)e^{-i2n\pi}\right) \\ &= \frac{1}{2N}\left(f(0) + 2\sum_{k=1}^{N-1} f(kh)e^{-i2nk\pi/N} + f(Nh)\right) \\ &= \frac{1}{N}\sum_{k=0}^{N-1} f(kh)e^{-i2nk\pi/N} \end{aligned}$$

Since

$$e^{-i2(N+n)k\pi/N} = e^{-i2k\pi - i2nk\pi/N} = e^{-i2k\pi}e^{-i2nk\pi/N} = e^{-i2nk\pi/N}$$

the estimate of c_{N+n} is the same as that for c_n, and there are only N independent estimates (as you would expect with only N sample points). Indeed, a sampling frequency of $2T/N$ is incapable of interpreting higher frequency responses, and the function f is represented by the estimate

$$f(t) \sim \frac{1}{N}\sum_{n=-\left[\frac{N-1}{2}\right]}^{\left[\frac{N}{2}\right]} C_n e^{in\pi t/T}$$

(i.e. if $N = 2P + 1$ the sum runs from $-P$ to P, while if $N = 2P$, it runs from $-(P-1)$ to P.)

where

$$C_n = \sum_{k=0}^{N-1} f(kh)e^{-i2nk\pi/N}\ .$$

At $t = rh = 2rT/N$, we have

$$\begin{aligned}
\frac{1}{N}\sum_{n=0}^{N-1} C_n e^{i2nr\pi/N} &= \frac{1}{N}\sum_{n=0}^{N-1}\sum_{k=0}^{N-1} f(kh) e^{i2n\pi(r-k)/N} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} f(kh) \sum_{n=0}^{N-1} e^{i2n\pi(r-k)/N} \\
&= \frac{1}{N}\sum_{k=0}^{N-1} f(kh) \left\{ \begin{array}{ll} N & r = k \\ 0 & r \neq k \end{array} \right. \quad = \quad f(rh)
\end{aligned}$$

so that the representation is exact at the sampling points.

The pair of equations

$$C_n = \sum_{k=0}^{N-1} f(kh) e^{-i2nk\pi/N}$$

$$f(rh) = \frac{1}{N} \sum_{n=-\left[\frac{N-1}{2}\right]}^{\left[\frac{N}{2}\right]} C_n e^{i2nr\pi/N}$$

or equivalently

$$f(rh) = \frac{1}{N}\sum_{n=0}^{N-1} C_n e^{i2nr\pi/N}$$

define the discrete Fourier transform and its inverse.

The transform has a straightforward matrix representation.

Setting $\omega = \exp(2\pi i/N)$, (so that $\omega^N = 1$,) we have

$$\begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ .. \\ C_{N-1} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^{-1} & \omega^{-2} & \dots & \omega^{-(N-1)} \\ 1 & \omega^{-2} & \omega^{-4} & \dots & \omega^{-(2N-2)} \\ .. & .. & .. & \dots & .. \\ 1 & \omega & \omega^2 & \dots & \omega^{N-1} \end{pmatrix} \begin{pmatrix} f(0) \\ f(h) \\ f(2h) \\ .. \\ f((N-1)h) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega^{N-1} & \omega^{N-2} & \dots & \omega \\ 1 & \omega^{N-2} & \omega^{N-4} & \dots & \omega^2 \\ .. & .. & .. & \dots & .. \\ 1 & \omega & \omega^2 & \dots & \omega^{N-1} \end{pmatrix} \begin{pmatrix} f(0) \\ f(h) \\ f(2h) \\ .. \\ f((N-1)h) \end{pmatrix}$$

$$\begin{pmatrix} f(0) \\ f(h) \\ f(2h) \\ .. \\ f((N-1)h) \end{pmatrix} = \frac{1}{N} \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 1 & \omega & \omega^2 & \dots & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & \dots & \omega^{N-2} \\ .. & .. & .. & \dots & .. \\ 1 & \omega^{N-1} & \omega^{N-2} & \dots & \omega \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ .. \\ C_{N-1} \end{pmatrix}$$

where we have used $\omega^N = 1$ to reduce the matrix coefficients to powers of ω between 0 and $N-1$.

Both the transform matrix and its inverse are symmetric.

While this transform is defined for complex values of $f(t)$, the function f is usually real, and the coefficients are given real interpretations.

Since $\omega^{-1} = \exp(-2\pi i/N) = \bar{\omega}$, when f is real $C_{N-n}(= C_{-n}) = \bar{C_n}$.

The contributions for n and $-n$ combine to give

$$\left(C_n e^{in\pi t/T} + \bar{C_n} e^{-in\pi t/T}\right)$$

which is real.

This expression has two standard representations.

Setting $C_n = A_n - iB_n$ gives

$$2A_n \cos\left(\frac{n\pi t}{T}\right) + 2B_n \sin\left(\frac{n\pi t}{T}\right)$$

while using the polar form $C_n = R_n e^{-i\phi_n}$ gives

$$2R_n \cos\left(\frac{n\pi t}{T} - \phi_n\right) \ .$$

When N is odd, the approximation for $f(t)$ can be written as

$$f(t) \sim \frac{1}{N} C_0 + \frac{2}{N} \sum_{n=1}^{(N-1)/2)} R_n \cos(n\pi t/T - \phi_n)$$

When N is even, C_n is real $(= A_n = R_n)$ when $n = N/2$, and the contibution to the sum is

$$R_n \cos\left(\frac{N\pi t}{2T}\right)$$

and

$$f(t) \sim \frac{1}{N} C_0 + \frac{2}{N} \sum_{n=1}^{N/2-1} R_n \cos(n\pi t/T - \phi_n) + \frac{1}{N} R_{N/2} \cos(N\pi t/2T)$$

For example, when $N = 3$ and $\omega = -\frac{1}{2} + i\frac{1}{2}\sqrt{3}$,

$$\begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix} \begin{pmatrix} f(0) \\ f(h) \\ f(2h) \end{pmatrix}$$

$$\begin{pmatrix} f(0) \\ f(h) \\ f(2h) \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \end{pmatrix}$$

$$\begin{aligned}
C_0 &= f(0) + f(h) + f(2h) \\
&= f(0) + f(h) + f(-h) \\
C_1 &= f(0) - \frac{1}{2}(f(h) + f(-h)) - i\frac{\sqrt{3}}{2}(f(h) - f(-h)) \\
C_2 &= f(0) - \frac{1}{2}(f(h) + f(-h)) + i\frac{\sqrt{3}}{2}(f(h) - f(-h)) \\
&= \bar{C}_1 \\
f(t) &\sim \frac{1}{3}(f(-h) + f(0) + f(h)) \\
&\quad + \frac{2}{3}\left(f(0) - \frac{1}{2}(f(-h) + f(h))\right)\cos\left(\frac{\pi t}{T}\right) \\
&\quad + \frac{1}{\sqrt{3}}(f(h) - f(-h))\sin\left(\frac{\pi t}{T}\right)
\end{aligned}$$

The other form can be obtained from this form.

Since

$$R\cos(\omega t - \phi) = R\cos(\phi)\cos(\omega t) + R\sin(\phi)\sin(\omega t)$$

we can write

$$A\cos(\omega t) + B\sin(\omega t)$$

in *Phase-Amplitude* form by equating

$$\begin{aligned}
A &= R\cos\phi \\
B &= R\sin\phi
\end{aligned}$$

This gives

$$\begin{aligned}
R &= \sqrt{A^2 + B^2} \\
\tan\phi &= \frac{B}{A}
\end{aligned}$$

where, as before, the appropriate value of ϕ is determined by considering one or other of the original equations.

Example.

If $f(-h) = 0$, $f(0) = 2$, and $f(h) = 1$,

$$\begin{aligned}
C_0 &= 2 + 1 + 0 = 3 \\
C_1 &= 2 + \left(-\frac{1}{2} - i\frac{\sqrt{3}}{3}\right) = \frac{3}{2} - i\frac{\sqrt{3}}{2} \\
C_2 &= 2 + \left(-\frac{1}{2} + i\frac{\sqrt{3}}{3}\right) = \frac{3}{2} + i\frac{\sqrt{3}}{2} \\
f(t) &\sim 1 + \cos\left(\frac{\pi t}{T}\right) + \frac{1}{\sqrt{3}}\sin\left(\frac{\pi t}{T}\right)
\end{aligned}$$

$$\sqrt{\left(1+\frac{1}{3}\right)} = \frac{2}{\sqrt{3}}$$

$$\tan\phi = \frac{1}{\sqrt{3}}\ ;\ \phi = \frac{\pi}{6}$$

$$f(t) \sim 1 + \frac{2}{\sqrt{3}}\cos\left(\frac{\pi t}{T} - \frac{\pi}{6}\right)$$

When $N = 4$ and $\omega = i$,

$$\begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \begin{pmatrix} f(0) \\ f(h) \\ f(2h) \\ f(3h) \end{pmatrix}$$
$$\begin{pmatrix} f(0) \\ f(h) \\ f(2h) \\ f(3h) \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & i & -1 & -i \\ 1 & -1 & 1 & -1 \\ 1 & -i & -1 & i \end{pmatrix} \begin{pmatrix} C_0 \\ C_1 \\ C_2 \\ C_3 \end{pmatrix}$$

$$\begin{aligned} C_0 &= f(0) + f(h) + f(2h) + f(3h) \\ C_1 &= (f(0) - f(2h)) - i(f(h) - f(3h)) \\ C_2 &= f(0) - f(h) + f(2h) - f(3h) \\ f(t) &\sim \frac{1}{4}(f(0) + f(h) + f(2h) + f(3h)) \\ &\quad + \frac{1}{2}(f(0) - f(2h))\cos\left(\frac{t\pi}{T}\right) + \frac{1}{2}(f(h) - f(3h))\sin\left(\frac{t\pi}{T}\right) \\ &\quad + \frac{1}{4}(f(0) - f(h) + f(2h) - f(3h))\cos\left(\frac{2t\pi}{T}\right) \end{aligned}$$

The matrices appearing in the second example can be factorised as

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -i \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & i \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & i & 0 & -i \end{pmatrix}$$

Factorisations of this type occur whenever $N = 2^r$. They are the basis of the **Fast Fourier Transform**, which is a computer implementation of the Discrete Fourier Transform.

Exercise.

The function $f(t)$ is periodic, with period 1.

If $f(0) = 0$, $f(0.2) = 0.6$, $f(0.4) = 2.0$, $f(0.6) = 2.2$ and $f(0.8) = 1.2$, determine the discrete Fourier Transformation of f, and determine the coefficients in the expansions

$$f(t) \sim a_0 + a_1 \cos(2\pi t) + b_1 \sin(2\pi t) + a_2 \cos(4\pi t) + b_2 \sin(4\pi t)$$
$$f(t) \sim c_0 + c_1 \cos(2\pi t - \phi_1) + c_2 \cos(4\pi t - \phi_2) \ .$$