Evaluation of Definite Integrals

Integrals of the type

$$\int_a^{a+2\pi} R(\cos\theta, \sin\theta)\, d\theta$$

where R is a rational function.

Consider $z = e^{i\theta}$. As θ increases from a to $a + 2\pi$, z traverses the unit circle once in the positive direction.

Also, if $z = e^{i\theta}$,

$$\begin{aligned}
\cos\theta &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \\
&= \frac{1}{2}\left(z + z^{-1}\right) = \frac{z^2+1}{2z} \\
\sin\theta &= \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right) \\
&= \frac{1}{2i}\left(z - z^{-1}\right) = \frac{z^2-1}{2iz} \\
dz &= ie^{i\theta}\, d\theta \\
d\theta &= \frac{dz}{iz}
\end{aligned}$$

Therefore

$$\begin{aligned}
&\int_a^{a+2\pi} R(\cos\theta, \sin\theta)\, d\theta \\
&= \oint_{|z|=1} R\left(\frac{z^2+1}{2z}, \frac{z^2-1}{2iz}\right)\frac{dz}{iz} \\
&= 2\pi \sum Res\left(\frac{1}{z}R\right)\Bigg|_{|z|<1}
\end{aligned}$$

e.g.

$$\int_{-\pi}^{\pi} \frac{\sin(2\theta)}{2+\cos\theta}\, d\theta$$

If $z = e^{i\theta}$,

$$\begin{aligned}
\sin(2\theta) &= \frac{1}{2i}\left(e^{2i\theta} - e^{-2i\theta}\right) \\
&= \frac{1}{2i}\left(z^2 - z^{-2}\right) = \frac{z^4-1}{2iz^2} \\
\frac{\sin(2\theta)}{2+\cos\theta} &= \frac{z^4-1}{2iz^2\left(2+\frac{z^2+1}{2z}\right)} \\
&= \frac{1}{i}\frac{z^4-1}{z^3+4z^2+z} \\
\int_{-\pi}^{\pi} \frac{\sin(2\theta)}{2+\cos\theta}\, d\theta &= \oint \frac{1-z^4}{z^2(z^2+4z+1)}\, dz
\end{aligned}$$

The integrand has poles at $z = 0$ (a pole of order 2) and when

$$z^2 + 4z + 1 = 0 \ , \quad z = -2 \pm \sqrt{3}$$

The residue at $z = 0$ is

$$\begin{aligned} &\lim_{z\to 0} \frac{d}{dz} \frac{1-z^4}{z^2+4z+1} \\ =\ &\lim_{z\to 0} \left(-\frac{4z^3}{z^2+4z+1} - \frac{(1-z^4)(2z+4)}{(z^2+4z+1)^2} \right) \\ =\ &-4 \end{aligned}$$

The residue at $z = -2 + \sqrt{3}$ is

$$\begin{aligned} &\left. \frac{1-z^4}{z^2(2z+4)} \right|_{-2+\sqrt{3}} \\ =\ &\frac{1-(97-56\sqrt{3})}{(7-4\sqrt{3})(2\sqrt{3})} \\ =\ &4 \end{aligned}$$

Since the sum of the residues inside the unit circle is 0,

$$\int_{-\pi}^{\pi} \frac{\sin(2\theta)}{2+\cos\theta} \, d\theta = 0$$

Integrals of the type

$$\int_{-\infty}^{\infty} f(x) \, dx$$

If $f(x)$ is a rational function,

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials, then the integral only exists if
(a) $q(x) \neq 0$ for $x \in \mathbb{R}$:
(b) $\partial(q(x)) \geq \partial(p(x)) + 2$

Consider

$$\oint_C f(z) \, dz$$

where C is the contour consisting of the real axis from $-R$ to R, combined with the semicircle $|z| = R$ in the upper half plane.

Let $M = \max|f|$ on the semicircle.

Since $\partial(q) \geq \partial(p) + 2$, $RM \to 0$ as $R \to \infty$.

Therefore

$$\left| \int_{|z|=R} f(z) \, dz \right| \leq \pi RM \downarrow 0 \text{ as } R \to \infty$$

Now, for sufficiently large R,

$$\oint_C f(z)\,dz = 2\pi i \sum_{Im(z)>0} Res\,(f(z))$$

Also,

$$\oint_C f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_{|z|=R} f(z)\,dz$$

so that, letting $R \to \infty$ we obtain

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{Im(z)>0} Res\,(f(z))$$

e.g.

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$$

$$\frac{1}{z^4+1}$$

has poles when $z^4 = -1$, $z = (\pm 1 \pm i)/\sqrt{2}$, of which $z = (\pm 1 + i)/\sqrt{2}$ are in the upper half plane.

The residue at z_k is

$$\frac{1}{4z_k^3} = \frac{z_k}{4z_k^4} = \frac{z_k}{-4}$$

Therefore

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+1}$$

$$= 2\pi i \left(-\frac{1}{4}\left(\frac{1+i}{\sqrt{2}}\right) - \frac{1}{4}\left(\frac{-1+i}{\sqrt{2}}\right)\right)$$

$$= 2\pi i \left(-\frac{i}{2\sqrt{2}} \right) = \frac{\pi}{\sqrt{2}}$$

This approach can also be used for integrals of the form

$$\int_0^{\infty} f(x)\,dx$$

PROVIDED THE INTEGRAND IS EVEN

that is

$$f(-x) = f(x)$$

In this case,

$$\int_0^{\infty} f(x)\,dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx$$

e.g.

$$\begin{aligned}\int_0^\infty \frac{x^2}{x^4+1}\,dx &= \frac{1}{2}\int_{-\infty}^\infty \frac{x^2}{x^4+1}\,dx \\ &= \pi i \sum_{Im(z)>0} Res\left(\frac{z^2}{z^4+1}\right) \\ &= \pi i \sum_{Im(z)>0} \left.\frac{z^2}{4z^3}\right|_{z_k} \\ &= \pi i\left(\frac{1}{2\sqrt{2}(1+i)} + \frac{1}{2\sqrt{2}(-1+i)}\right) \\ &= \pi i\left(\frac{1-i}{4\sqrt{2}} + \frac{-1-i}{4\sqrt{2}}\right) = \frac{\pi}{2\sqrt{2}}\end{aligned}$$

If the integrand is not even, we can still use residues to calculate $\int_0^\infty f(x)\,dx$ where $f(z)$ is singlevalued in $\mathbb{C}$, and $|z||f(z)| \to 0$ as $|z| \to \infty$.
However, in this case we use the multivalued property of the logarithm function.

The contour used for integrals of this type is called a *keyhole contour*.

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It consists of the following pieces:

(a) The axis $z = x$ from $x = \epsilon$ to $x = R$

(b) The circle $z = Re^{i\theta}$ from $\theta = 0$ to $\theta = 2\pi$

(c) The axis $z = xe^{2\pi i}$ from $x = R$ to $x = \epsilon$

(d) The circle $z = \epsilon e^{i\theta}$ from $\theta = 2\pi$ to $\theta = 0$

We consider

$$\oint f(z)\log z\,dz$$

Since $\log z$ is regular inside and on the contour,

$$\oint f(z)\log z\,dz = 2\pi i \sum Res f(z)\log z$$

where the residues are evaluated at the poles of f.

While the function f is singlevalued and can be handled without difficulty, it is important to remember when evaluating the log function that $0 \le \arg(z) \le 2\pi$.

Now consider the form of the integral on the various parts of the contour.

On (a) : $z = x$, $\log z = \log x$

$$\int f(z)\log z\,dz = \int_\epsilon^R f(x)\log x\,dx$$

On (b) : $z = Re^{i\theta}$, $|f(z)| = O(R^{-2})$, $|\log z| = O(\log R)$
The arc length is $2\pi R$, so that

$$\left|\int f(z)\log z\,dz\right| = O(R.R^{-2}\log R) = O(R^{-1}\log R)$$

which goes to 0 as $R \to \infty$.

On (c) : $z = xe^{2\pi i}$, $f(z) = f(x)$, $dz = dx$,
$\log z = \log x + 2\pi i$.

$$\begin{aligned}\int f(z)\log z\,dz &= \int_R^\epsilon f(x)(\log x + 2\pi i)\,dx \\ &= -\int_\epsilon^R f(x)\log x\,dx - 2\pi i\int_\epsilon^R f(x)\,dx\end{aligned}$$

On (d) : $z = \epsilon e^{i\theta}$, $|f(z)|$ is bounded, $|\log z| = O(\log\epsilon)$
The arc length is $2\pi\epsilon$, so that

$$\left|\int f(z)\log z\,dz\right| = O(\epsilon\log\epsilon)$$

which goes to 0 as $\epsilon \to 0$.

Therefore

$$\oint f(z)\log z\,dz = \ 2\pi i\sum Res(f(z)\log z)$$
$$= \ -2\pi i\int_\epsilon^R f(x)\,dx + O(R^{-1}\log R) + O(\epsilon\log\epsilon)$$

Taking the limits we obtain

$$\int_0^\infty f(x)\,dx = -\sum Res(f(z)\log z)$$

where $0 \le \arg(z) \le 2\pi$

e.g.

$$\int_0^\infty \frac{dx}{1+x^3}$$

$f(z) = 1/(1+z^3)$ has poles when $z^3 = -1$.

$$z^3 = e^{(2n+1)\pi i}\ ;\ z = e^{\pi i/3}\ ,\ e^{\pi i}\ ,\ e^{5\pi i/3}$$

The residues are

$$\frac{1}{3z^2}\log z = -\frac{1}{3}z\log z$$

At $z = e^{\pi i/3}$:

$$-\frac{1}{3}\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\left(\frac{\pi i}{3}\right) = \frac{\pi}{6\sqrt{3}} - i\frac{\pi}{18}$$

At $z = e^{\pi i}$:

$$-\frac{1}{3}(-1)(\pi i) = i\frac{\pi}{3}$$

At $z = e^{5\pi i/3}$:

$$-\frac{1}{3}\left(\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\left(\frac{5\pi i}{3}\right) = -\frac{5\pi}{6\sqrt{3}} - i\frac{5\pi}{18}$$

The sum of the residues is

$$-\frac{4\pi}{6\sqrt{3}} + i\pi\left(-\frac{1}{18}+\frac{1}{3}-\frac{5}{18}\right) = -\frac{2\pi}{3\sqrt{3}}$$

Therefore

$$\int_0^\infty \frac{dx}{1+x^3} = \frac{2\pi}{3\sqrt{3}}$$

The keyhole contour can also be used for functions which already have branch points at the origin.

e.g.

$$\int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx$$

Consider

$$\oint_C \frac{z^{1/2}}{1+z^2}\,dz$$

where C is the keyhole contour.

The function $z^{1/2}$ has a branch point at $z = 0$, so that we have to take the argument of z into account when evaluating the component parts.

on (a): $z = x$, $z^{1/2} = x^{1/2}$.

The contibution to the integral is

$$\int_\epsilon^R \frac{x^{1/2}}{1+x^2}\,dx$$

on (b): $z = Re^{i\theta}$, $|z^{1/2}/(1+z^2)| = O(R^{-3/2})$.

Since the path length is $2\pi R$,

$$\left|\int \frac{z^{1/2}}{1+z^2}\,dz\right| = O(R^{-1/2})$$

on (c): $z = xe^{2\pi i}$, $z^{1/2} = x^{1/2}e^{\pi i} = -x^{1/2}$.

The contribution to the integral is

$$\int_R^\epsilon \frac{-x^{1/2}}{1+x^2}\,dx = \int_\epsilon^R \frac{x^{1/2}}{1+x^2}\,dx$$

on (d): $z = \epsilon e^{i\theta}$, $|z^{1/2}/(1+z^2)| = O(\epsilon^{1/2})$.
Since the path length is $2\pi\epsilon$,

$$\left|\int \frac{z^{1/2}}{1+z^2}\,dz\right| = O(\epsilon^{3/2})$$

On the other hand

$$\oint_C \frac{z^{1/2}}{1+z^2}\,dz = 2\pi i \sum Res$$

The integrand has poles at $z = \pm i$. Since the argument is important when evaluating $z^{1/2}$, these points need to be expressed as $z = e^{\pi i/2}$ and $z = e^{3\pi i/2}$.

The residues are found by evaluating $z^{1/2}/(2z)$ at the poles.
This gives:

$$\text{at} \quad z = e^{\pi/2}\ ;\ \frac{1}{2i}e^{\pi/4} = \frac{1}{2i}\left(\frac{1+i}{\sqrt{2}}\right)$$

$$\text{at} \quad z = e^{3\pi/2}\ ;\ -\frac{1}{2i}e^{3\pi/4} = -\frac{1}{2i}\left(\frac{-1+i}{\sqrt{2}}\right)$$

$$\oint_C \frac{z^{1/2}}{1+z^2}\,dz = 2\pi i\frac{1}{2i}\sqrt{2} = \sqrt{2}\pi$$

Therefore

$$2\int_\epsilon^R \frac{x^{1/2}}{1+x^2}\,dx + O(R^{-1/2}) + O(\epsilon^{3/2}) = \sqrt{2}\pi$$

Taking the limits, and dividing by 2 gives

$$\int_0^\infty \frac{x^{1/2}}{1+x^2}\,dx = \frac{\pi}{\sqrt{2}}$$

Finally, we consider integrals of the type which occur in the Fourier Integral Transform.

We begin by stating the result known as *Jordan's Lemma*.

If $|zf(z)|$ is bounded as $|z| \to \infty$, then

(a) if $a > 0$,

$$\int_C f(z)e^{iaz}\,dz \to 0$$

as $R \to \infty$, where C is the semicircle $z = Re^{i\theta}$, $0 \le \theta \le \pi$;

(b) if $a < 0$,

$$\int_C f(z)e^{iaz}\,dz \to 0$$

as $R \to \infty$, where C is the semicircle $z = Re^{i\theta}$, $0 \ge \theta \ge -\pi$.

e.g.

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x^2+1}\,dx$$

For $\alpha > 0$, we can consider

$$\oint_C \frac{e^{i\alpha z}}{z^2+1}\,dz$$

where C is the contour consisting of the real axis from $-R$ to R combined with the semicircle $z = Re^{i\theta}$, $0 \leq \theta \leq \pi$.

If $R > 1$, there is one pole inside the contour at $z = i$, at which the residue is

$$\left.\frac{e^{i\alpha z}}{2z}\right|_{z=i} = \frac{e^{-\alpha}}{2i}$$

Hence

$$\oint_C \frac{e^{i\alpha z}}{z^2+1}\,dz = (2\pi i)\frac{e^{-\alpha}}{2i} = \pi e^{-\alpha}$$

Now, letting $R \to \infty$, Jordan's Lemma ensures that the contribution to the contour integral from the semicircular path goes to 0, so that

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x^2+1}\,dx = \pi$$

For $\alpha < 0$, we can consider

$$\oint_C \frac{e^{i\alpha z}}{z^2+1}\,dz$$

where C is the contour consisting of the real axis from $-R$ to R combined with the semicircle $z = Re^{i\theta}$, $0 \geq \theta \geq -\pi$.

Note that this contour is traversed in the clockwise - negative - direction.

If $R > 1$, there is one pole inside the contour at $z = -i$, at which the residue is

$$\left.\frac{e^{i\alpha z}}{2z}\right|_{z=-i} = \frac{e^{\alpha}}{-2i}$$

Hence

$$\oint_C \frac{e^{i\alpha z}}{z^2+1}\,dz = (-2\pi i)\frac{e^{\alpha}}{-2i} = \pi e^{\alpha}$$

where the negative sign arises from the direction in which the contour is traversed.

Now, letting $R \to \infty$, the contribution to the contour integral from the semicircular path again goes to 0, so that

$$\int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x^2+1}\,dx = \pi e^{\alpha} \quad \text{when} \quad \alpha < 0$$

e.g.

$$\int_0^\infty \frac{\cos 2\pi x}{x^4+x^2+1}\,dx$$

$$= \frac{1}{2}\int_{-\infty}^\infty \frac{\cos 2\pi x}{x^4+x^2+1}\,dx$$

because the integrand is even,

$$= \frac{1}{2} Re\left(\int_{\infty}^\infty \frac{e^{2\pi i x}}{x^4+x^2+1}\,dx\right)$$

(This expression for $\cos 2\pi x$ allows us to make use of Jordan's Lemma. The value of the integral is therefore $2\pi i$ times the sum of the residues in the upper half plane.)

$$= Re\left(\pi i \sum Res\left(\frac{e^{2\pi i z}}{z^4+z^2+1}\right)\bigg|_{y>0}\right)$$

$$z^4+z^2+1 = (z^6-1)/(z^2-1)$$

so that the zeros of the denomianator occur when

$$z = e^{\pi i/3}\ ,\ e^{2\pi i/3}\ ,\ e^{4\pi i/3} \text{ and } e^{5\pi i/3}$$

Of these, the first two lie in the upper half plane at

$$z = \pm\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

The residues are found by evaluating

$$\frac{e^{2\pi i z}}{4z^3+2z}$$

When

$$z = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$z^3 = -1\ ;\ 2z = 1+\sqrt{3}i$$

$$\frac{e^{2\pi i z}}{4z^3+2z} = \frac{e^{\pi i}e^{-\sqrt{3}\pi}}{-3+\sqrt{3}i}$$

$$= -\frac{e^{-\sqrt{3}\pi}}{-3+\sqrt{3}i}\,\frac{-3-\sqrt{3}i}{-3-\sqrt{3}i} = \frac{3+\sqrt{3}i}{12}e^{-\sqrt{3}\pi}$$

When

$$z = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

$$z^3 = 1\ ;\ 2z = -1+\sqrt{3}i$$

$$\frac{e^{2\pi i z}}{4z^3+2z} = \frac{e^{-\pi i}e^{-\sqrt{3}\pi}}{3+\sqrt{3}i}$$

$$= -\frac{e^{-\sqrt{3}\pi}}{3+\sqrt{3}i}\,\frac{3-\sqrt{3}i}{3-\sqrt{3}i} = \frac{-3+\sqrt{3}i}{12}e^{-\sqrt{3}\pi}$$

The sum of the residues is

$$\frac{\sqrt{3}i}{6}e^{-\sqrt{3}\pi}$$

so that

$$\int_0^{\infty}\frac{\cos 2\pi x}{x^4+x^2+1}\,dx=-\frac{\pi}{2\sqrt{3}}e^{-\sqrt{3}\pi}$$