The mapping $w = \sin z$

Consider firstly the strip $-\frac{\pi}{2} < x < \frac{\pi}{2}$ in the $z-$plane.

If $w = u + iv + \sin(x + iy)$, then

$$\begin{aligned} u = &\ \sin x \cosh y \\ v = &\ \cos x \sinh y \end{aligned}$$

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The infinite lines x constant,

$$-\infty < y < \infty$$

map as follows;

If $x = 0$, $u = 0$ and $v = \sinh y$, so that as y varies from ∞ to ∞, the imaginary $y-$axis in the $z-$plane maps onto the imaginary $v-$axis in the $w-$plane, traversed from $-\infty$ to ∞.

If $x = \frac{\pi}{2}$, $u = \cosh y$ and $v = 0$.

As y varies from $-\infty$ to ∞, w varies from ∞ to 1 (when $y = 0$) and then back to ∞ along the positive real $u-$axis. To separate these two line segements, we introduce a cut from 1 to ∞ along this positive real axis.

When $x = \frac{\pi}{2}-$, $\cos x > 0$, so that $v < 0$ when $y < 0$. Hence the line segment $y < 0$ corresponds to the lower edge of the cut, and the segment $y > 0$ to the upper edge.

When $x = -\frac{\pi}{2}$, $u = -\cosh y$ and $v = 0$.

In this case, as y varies from $-\infty$ to ∞, w varies from $-\infty$ to -1 and back along the negative real $u-$axis. We separate these line segments by introducing a cut from $-\infty$ to -1 along the negative real axis.

When $x = -\frac{\pi}{2}+$, $\cos x > 0$, so that $v < 0$ when $y < 0$. Hence the line segment $y < 0$ corresponds to the lower edge of the cut, and the segment $y > 0$ to the upper edge.

In general, when $x = c$

$$u = \sin c \cosh y \ ; \ \cosh y = \frac{u}{\sin c}$$
$$v = \cos c \sinh y \ ; \ \sinh y = \frac{v}{\cos c}$$
$$\frac{u^2}{\sin^2 c} - \frac{v^2}{\cos^2 c} = \cosh^2 y - \sinh^2 y = 1$$

This represents a branch of a hyperbola in the $w-$**plane.**

Since $\cos c > 0$ **for** $-\frac{\pi}{2} < c < \frac{\pi}{2}$**, the branch is traversed from the lower half plane to the upper half plane as** y **ranges from** $-\infty$ **to** ∞**.**

Since $\sin^2(-c) = \sin^2 c$ **and** $\cos^2(-c) = \cos^2 c$**,** c **and** $-c$ **correspond to the two branches of the same hyperbola. When** $c > 0$**,** $\sin c > 0$**, so that** $u = \sin c \cosh y > 0$**, and we have the branch in the right half plane, while when** $c < 0$ **we have the left hand branch.**

All these hyperbolas have the same foci at $u = 1$ **and** $u = -1$**.**

The line segments $y = c$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$:

When $y = 0$, $u = \sin x$ and $v = 0$, so that the line segment $-\frac{\pi}{2} < x < \frac{\pi}{2}$ maps onto the line segment $-1 < u < 1$ on the $u-$axis.

Otherwise, we have

$$u = \sin x \cosh c \ ; \ \sin x = \frac{u}{\cosh c}$$

$$v = \cos x \sinh c \ ; \ \cos x = \frac{v}{\sinh c}$$

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = \sin^2 x + \cos^2 x = 1$$

Hence the line segments map onto semi-ellipses. If $c > 0$, the semi-ellipse corresponding to $y = c$ lies in the upper half plane, while the other half of the same ellipse corresponds to $y = -c$.

These two elliptic segments are separated by the branch cuts along the positive and negative axes.

Once again, all these ellipses have the same foci; $u = -1$ and $u = 1$.

The far field

For large values of the argument,

$$\cosh y \sim \sinh y \sim \frac{1}{2}e^{y}$$

Therefore, when we are a long way from the origin, the hyperbolic segment

$$\begin{aligned} u &= \sin c \cosh y \\ v &= \cos c \sinh y \end{aligned}$$

is asymptotic to the straight line $v = (\cot c)\, u$.

Similarly, the elliptic segment

$$\frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1$$

becomes like the circular segment

$$u^2 + v^2 = \frac{1}{4}e^{2c}$$

Therefore, the far field looks like line segments originating from the origin, and circles centred at the origin.

Extending the mapping

If we consider the strip $\frac{\pi}{2} < x < \frac{3\pi}{2}$, we find that it also maps onto a plane cut from $-\infty$ to -1 and from 1 to ∞ along the real $u-$axis.

However, in this case we have

$$v = \cos x \sinh y$$

where $\cos x < 0$, so that as y ranges from $-\infty$ to ∞, v goes from ∞ to $-\infty$.

Similarly, on the cut from 1 to ∞ along the $u-$axis which corresponds to $x = \frac{\pi}{2}$, the upper edge corresponds to $y < 0$, and the lower edge to $y > 0$.

Therefore this sheet of the Riemann surface joins with the first one by interleaving (in the fourth dimension) across the cut along the positive axis.

In turn we join this sheet to that corresponding to the strip $\frac{3\pi}{2} < x < \frac{5\pi}{2}$ by interleaving across the cut along the negative axis.

Note that, since $\sin z$ has period 2π, the strip $\frac{3\pi}{2} < x < \frac{5\pi}{2}$ maps in the same fashion as the strip $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

The total Riemann surface corresponding to this mapping is obtained by repeating this process indefinitely. The surface consists of an infinite number of planes which are joined by interleavings along the positive and negative real axes alternatively.

Hyperbolic Functions

$$\cosh z = \frac{e^z + e^{-z}}{2}$$
$$\sinh z = \frac{e^z - e^{-z}}{2}$$

$$\cosh(iz) = \frac{e^{iz} + e^{-iz}}{2} = \cos z$$
$$\sinh(iz) = \frac{e^{iz} - e^{-iz}}{2} = i \sin z$$

$$\cos(iz) = \cosh(i^2 z) = \cosh(-z) = \cosh z$$
$$i \sin(iz) = \sinh(i^2 z) = \sinh(-z) = -\sinh z$$
$$\sin(iz) = i \sinh(z)$$

The hyperbolic functions are periodic with period $2\pi i$

$\sinh z = 0$ **when** $\sin(iz) = 0$
$\sin(iz) = 0$ **when** $iz = n\pi$**,** $z = -in\pi$**.**
Similarly, $\cosh z = 0$ **when** $z = i(n + \frac{1}{2})\pi$**.**

$$\sinh(x + iy) = -i\sin(i(x + iy)) = -i\sin(-y + ix)$$
$$= -i(\sin(-y)\cosh(x) + i\cos(-y)\sinh(x))$$
$$= \sinh x \cos y + i\cosh x \sin y$$

$$|\sinh z|^2 = \sinh^2 x \cos^2 y + \cosh^2 x \sin^2 y$$
$$= \sinh^2 x \cos^2 y + (1 + \sinh^2 x)\sin^2 y$$
$$= \sinh^2 x(\cos^2 y + \sin^2 y) + \sin^2 y$$
$$= \sinh^2 x + \sin^2 y$$

Similarly

$$\cosh z = \cosh x \cos y + i\sinh x \sin y$$

$$|\cosh z|^2 = \sinh^2 x + \cos^2 y$$