Application of Conformal Mapping to the solution of Laplace's Equation

Suppose that the regular function $w = f(z)$ maps the region $\mathcal{D}$ conformally onto the region $\mathcal{D}'$.

If $H(u, v)$ is a harmonic function on $\mathcal{D}'$, i.e.

$$\frac{\partial^2 H}{\partial u^2} + \frac{\partial^2 H}{\partial v^2} = 0$$

then there is a conjugate harmonic function $J(u, v)$ such that

$$F(w) = H + iJ$$

is regular on $\mathcal{D}'$.

It follows that the function

$$F(f(z)) = H(u(x, y), v(x, y)) + iJ(u(x, y), v(x, y))$$

is regular on $\mathcal{D}$.

Therefore

$$H(u(x, y), v(x, y))$$

is a harmonic function on $\mathcal{D}$.

e.g. For $F(w) = e^w$,

$$H(u, v) = e^u \cos v$$

If we take $w = z^2 = x^2 - y^2 + 2ixy$, we obtain

$$H(u(x, y), v(x, y)) = e^{x^2 - y^2} \cos(2xy)$$

$$\begin{aligned}
H_x &= 2xe^{x^2-y^2} \cos(2xy) \\
&\quad -2ye^{x^2-y^2} \sin(2xy) \\
H_{xx} &= (2 + 4x^2)e^{x^2-y^2} \cos(2xy) \\
&\quad -8xye^{x^2-y^2} \sin(2xy) \\
&\quad -4y^2e^{x^2-y^2} \cos(2xy) \\
H_y &= -2ye^{x^2-y^2} \cos(2xy) \\
&\quad -2xe^{x^2-y^2} \sin(2xy) \\
H_{yy} &= (-2 + 4y^2)e^{x^2-y^2} \cos(2xy) \\
&\quad +8xye^{x^2-y^2} \sin(2xy) \\
&\quad -4x^2e^{x^2-y^2} \cos(2xy) \\
H_{xx} + H_{yy} &= 0
\end{aligned}$$

Suppose that $\mathcal{C} = \{X(t), Y(t)\}$ is a curve in $\mathcal{D}$ along which

$$H(u(x, y), v(x, y)) = c$$

where c is a constant.

If $w = f(z)$ maps $\mathcal{C}$ onto $\mathcal{C}' = \{U(t), V(t)\}$ in $\mathcal{D}'$, where

$$U(t) = u(X(t), Y(t))$$
$$V(t) = v(X(t), Y(t))$$

then on $\mathcal{C}'$ we have

$$H(U(t), V(t)) = H(u(X(t), Y(t)), v(X(t), Y(t))) = c$$

In particular, if $\mathcal{C}$ is part of the boundary of $\mathcal{D}$, then $\mathcal{C}'$ is part of the boundary of $\mathcal{D}'$, and we see that the *Dirichlet boundary condition*

$$H = c$$

is invariant under the conformal mapping.

Similarly, if the conjugate function

$$J(u(x, y), v(x, y))$$

is a constant along a smooth line $\mathcal{C}$, then

$$J(u, v)$$

is constant along $\mathcal{C}'$.

But along these lines the tangential derivative

$$\frac{\partial J}{\partial t} = 0$$

and therefore, by the Cauchy-Riemann equations, the normal derivative

$$\frac{\partial H}{\partial n} = 0$$

This means that the *Neumann boundary condition*

$$\frac{\partial H}{\partial n} = 0$$

is also invariant under conformal mapping.

e.g.

$$H(x,y) = \frac{2}{\pi}x$$

is a solution of Laplace's equation on the strip $\mathcal{D}$

$$-\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$y > 0$$

which satisfies the boundary conditions

$$H = -1 \ ; \ x = -\frac{\pi}{2}$$
$$H = 1 \ ; \ x = \frac{\pi}{2}$$
$$\frac{\partial H}{\partial n} = \frac{\partial H}{\partial y} = 0 \ ; \ y = 0$$

H is the real part of the function $f(z) = \frac{2}{\pi}z$.

The mapping

$$w = \sin z$$

maps $\mathcal{D}$ onto the upper half plane, so that

(a) The line segment

$$x = -\frac{\pi}{2} \ ; \ y \geq 0$$

maps onto the line segment

$$v = 0 \ ; \ u \leq -1$$

(b) The line segment

$$x = \frac{\pi}{2} \ ; \ y \geq 0$$

maps onto the line segment

$$v = 0 \ ; \ u \geq 1$$

(c) the line segment

$$y = 0 \ ; -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

maps onto the line segment

$$v = 0 \ ; \ -1 \leq u \leq 1$$

Hence the mapping defines the solution of Laplace's equation

$$H_{uu} + H_{vv} = 0$$

in the region $v > 0$, for which

$$H(u,0) = -1 \ ; \ u < -1$$
$$\frac{\partial H}{\partial v}(u,0) = 0 \ ; \ -1 < u < 1$$
$$H(u,0) = 1 \ ; \ u > 1$$

When

$$
\begin{aligned}
w &= \sin z \\
z &= \frac{1}{i}\log(iw + \sqrt{1-w^2}) \\
x &= \arg(iw + \sqrt{1-w^2}) \\
H &= \frac{2}{\pi}x \\
&= \frac{2}{\pi}\arg(-v + iu + \sqrt{1-u^2+v^2-2iuv})
\end{aligned}
$$

Consider the problem of finding the solution of Laplace's equation in the region between two non-intersecting circles $\mathcal{C}_1$ and $\mathcal{C}_2$ which takes the constant value a on $\mathcal{C}_1$ and the constant value b on $\mathcal{C}_2$.

There is a Möbius transformation which maps these circles onto concentric circles

$$\begin{aligned} \mathcal{C}_1' &:= \ |w| = \rho \\ \mathcal{C}_2' &:= |w| = 1 \end{aligned}$$

The problem is now radially symmetric, so that the solution of Laplace's equation is a solution in the radial variable alone.

The most general such solution is

$$H = \alpha + \beta \log |w|$$

Substituting the boundary conditions, we obtain

$$\begin{aligned} a &= \alpha + \beta \log \rho \\ b &= \alpha + \beta \log 1 \ = \ \alpha \\ \beta &= \frac{a-b}{\log \rho} \\ H &= b + \frac{a-b}{\log \rho} \log |w| \end{aligned}$$

Green's Functions

A special solution of Laplace's equation in a region $\mathcal{D}$ is the *Green's function*. This is a solution $G(z, z_0)$ which has the properties

(a) The solution has a logarithmic singularity

$$G(z, z_0) \sim -\frac{1}{2\pi}\log(|z - z_0|)$$

at an arbitrary fixed point z_0 in $\mathcal{D}$.

(b) $G(z, z_0) = 0$ when z is on the boundary of $\mathcal{D}$.

Where the region $\mathcal{D}$ is simply connected, we can determine G by finding a function $w = f(z, z_0)$ which maps $\mathcal{D}$ onto the unit circle in the w plane, so that z_0 maps onto $w = 0$.

(There is a theorem; *the Riemann Mapping Theorem*; which shows that there is always such a function.)

In the transformed region we want a harmonic function such that

(a)
$$G \sim -\frac{1}{2\pi}\log|w|$$

(b)
$$G = 0 \text{ when } |w| = 1$$

This solution is obviously

$$G = -\frac{1}{2\pi}\log|w|$$

so that the Green's function is

$$G(z, z_0) = -\frac{1}{2\pi}\log|f(z, z_0)|$$

e.g. For the upper half plane, the Möbius transformation

$$w = \frac{z - z_0}{z - z_0^*}$$

maps the upper half plane onto the unit circle, and maps z_0 onto the origin.

Therefore the Green's Function for the upper half plane is

$$\begin{aligned} G(z, z_0) = & -\frac{1}{2\pi}\log\left|\frac{z - z_0}{z - z_0^*}\right| \\ = & -\frac{1}{4\pi}\log\left(\frac{(x - x_0)^2 + (y - y_0)^2}{(x - x_0)^2 + (y + y_0)^2}\right) \end{aligned}$$

For the unit circle, the Möbius transformation

$$w = \frac{z - z_0}{1 - zz_0^*}$$

maps the unit circle onto itself, and maps z_0 onto the origin.

Therefore the Green's function for the unit circle is

$$\begin{aligned} G(z, z_0) = & -\frac{1}{2\pi}\log\left|\frac{z - z_0}{1 - zz_0^*}\right| \\ = & -\frac{1}{4\pi}\log\left(\frac{(x - x_0)^2 + (y - y_0)^2}{(1 - xx_0 - yy_0)^2 + (xy_0 - yx_0)^2}\right) \end{aligned}$$