$$
\begin{aligned}
f(x) &= \frac{\sin bx}{x} \qquad (b > 0) \\
\mathcal{F}(\omega) &= \int_{-\infty}^{\infty} \frac{\sin bx}{x} e^{-i\omega x}\, dx
\end{aligned}
$$

If we write

$$\sin bx = \frac{1}{2i}\left(e^{ibx} - e^{-ibx}\right)$$

we get

$$\mathcal{F}(\omega) = \frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{i(b-\omega)x}}{x}\, dx \; - \; \frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{-i(b+\omega)x}}{x}\, dx \; .$$

Unfortunately, neither integral exists.

We need to modify the integral before we make the substitution.

$$
\begin{aligned}
\mathcal{F}(\omega) &= \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin bx}{x} e^{-i\omega x}\, dx \\
&= \lim_{R\to\infty} \int_{-R}^{R} \frac{\sin bz}{z} e^{-i\omega z}\, dz
\end{aligned}
$$

Since

$$\frac{\sin bz}{z} e^{-i\omega z}$$

is regular (it is entire) it is holomorphic, so that considered as a line integral in $\mathbb{C}$ its value is path independent. Hence we can choose a new path between $-R$ and R which avoids the origin.

We can now safely write

$$
\begin{aligned}
&\int_{-R}^{R} \frac{\sin bz}{z} e^{-i\omega z}\, dz \\
&= \frac{1}{2i}\int_{-R}^{R} \frac{e^{i(b-\omega)z}}{z}\, dz \; - \; \frac{1}{2i}\int_{-R}^{R} \frac{e^{-(b+\omega)z}}{z}\, dz
\end{aligned}
$$

Both these integrals have the form

$$\int_{-R}^{R} \frac{e^{iaz}}{z}\, dz \; .$$

For $a > 0$, we close the path of integration by using a semicircle in the upper half plane.

By Jordan's lemma, the integral over this section is $O(1/R)$.

Because we have chosen to pass below the origin, there is one pole of the integrand inside the contour at $z = 0$, at which the residue is 1.

Therefore, by the residue theorem,

$$\int_{-R}^{R} \frac{e^{iaz}}{z}\,dz = 2\pi i + O(1/R) \quad a > 0$$

or

$$\int_{-\infty}^{\infty} \frac{e^{iaz}}{z}\,dz = 2\pi i \quad a > 0$$

For $a < 0$, we close the path of integration by using a semicircle in the lower half plane.

Since there are no singularities in the lower half plane, the value of the resulting integral is 0 and therefore

$$\int_{-R}^{R} \frac{e^{iaz}}{z}\,dz = 0 + O(1/R) \quad a < 0$$

or

$$\int_{-\infty}^{\infty} \frac{e^{iaz}}{z}\,dz = 0 \quad a < 0$$

Returning to our Fourier transform, we find that

$$\frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{i(b-\omega)z}}{z}\,dz = \begin{cases} \pi & b > \omega \\ 0 & b < \omega \end{cases}$$

$$\frac{1}{2i}\int_{-\infty}^{\infty} \frac{e^{-i(b+\omega)z}}{z}\,dz = \begin{cases} \pi & -b > \omega \\ 0 & -b < \omega \end{cases}$$

$$\mathcal{F}(\omega) = \begin{cases} 0 & \omega < -b \\ \pi & -b < \omega < b \\ 0 & \omega > b \end{cases}$$

Consider the following problem:

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$
$$u(x,0) = 0 \ ; \ u_t(x,0) = 0$$
$$u(0,t) = 0 \ ; \ u(\pi,t) = 1$$

We can solve this problem by taking a Laplace transform with respect to the variable t.

Let

$$U(x,p) = \int_0^{\infty} e^{-pt} u(x,t)\,dt \ .$$

Then

$$\int_0^{\infty} e^{-pt} u_{tt}(x,t)\,dt = p^2 U(x,p)$$

and

$$\int_0^{\infty} e^{-pt} u_{xx}(x,t)\,dt = U_{xx}(x,p) \ .$$

From the boundary conditions we obtain

$$U(0,p) = \int_0^\infty e^{-pt} u(0,t)\,dt = 0$$
$$U(\pi,p) = \int_0^\infty e^{-pt} u(\pi,t)\,dt = \frac{1}{p}$$

The function U therefore satisfies the ordinary differential equation

$$U_{xx} = p^2 U$$
$$U(0,p) = 0\ ,\ U(\pi,p) = \frac{1}{p}$$

$$U(x,p) = A\sinh(px) + B\cosh(px)$$
$$U(0,p) = B = 0$$
$$U(\pi,p) = A\sinh(p\pi) = \frac{1}{p}$$
$$A = \frac{1}{p\sinh(p\pi)}$$
$$U(x,p) = \frac{1}{p}\frac{\sinh(px)}{\sinh(p\pi)}$$

From the inversion formula, we have

$$u(x,t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{1}{p}\frac{\sinh(px)}{\sinh(p\pi)} e^{pt}\,dp$$

where $c > 0$.

All the poles of the integrand lie on the imaginary axis, at $p = 0$ and at the zeros of $\sinh(p\pi)$; $p = in$.

At $p = 0$, the zeros of $\sinh(px)$ and $\sinh(p\pi)$ cancel each other out, so that we have a simple pole, and the residue is

$$\lim_{p\to 0}\frac{\sinh(px)}{\sinh(p\pi)} = \frac{x}{\pi}$$

while at $p = in$, $n = \pm 1, \pm 2, \ldots$, the residues are

$$\left.\frac{\sinh(px)}{p} e^{pt} \frac{1}{\pi\cosh(p\pi)}\right|_{p=in}$$
$$= \frac{\sinh(inx)}{in} e^{int} \frac{1}{\pi\cosh(in\pi)}$$
$$= \frac{\sin(nx)}{n} e^{int} \frac{1}{\pi\cos(n\pi)} = (-1)^n \frac{\sin(nx)}{n\pi} e^{int}$$

However, there are infinitely many poles, therefore we need to take care when closing the contour on the left.

If we choose the arc $|p| = N + \frac{1}{2}$, then $\sinh(px)/\sinh(p\pi)$ is bounded on this curve, and by Jordan's lemma

$$\int \frac{1}{p}\frac{\sinh(px)}{\sinh(p\pi)} e^{pt}\,dp = O(1/|p|) = O(1/N)$$

along this path.

$$\begin{aligned}
&\frac{1}{2\pi i}\oint \frac{1}{p}\frac{\sinh(px)}{\sinh(p\pi} e^{pt}\,dp \\
&= \frac{1}{2\pi i}\int_{c-(N+1/2)i}^{c+(N+1/2)i} \frac{1}{p}\frac{\sinh(px)}{\sinh(p\pi)} e^{pt}\,dp + O(1/N) \\
&= \frac{x}{\pi} + \sum_{n=1}^{N} (-1)^n \frac{\sin(nx)}{n\pi}\left(e^{int} + e^{-int}\right) \\
&= \frac{x}{\pi} + \sum_{n=1}^{N} \frac{(-1)^n 2}{n\pi} \sin(nx)\cos(nt)
\end{aligned}$$

Now, taking the limit as $N \to \infty$,

$$u(x,t) = \frac{x}{\pi} + \frac{2}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin(nx)\cos(nt)\ .$$